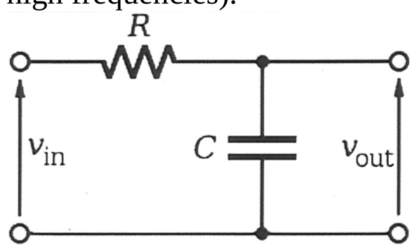
The Hemmi 301 and Boonshaft & Fuchs Slide Rules
A Brief Introduction to Transfer Functions

Richard Smith Hughes

This brief introduction on transfer functions presents the basic equations discussed in my JOS Hemmi 301 and Boonshaft & Fuchs article. The basic theory is really quite simple, basically just solving for the hypotenuse and phase angle of a right triangle. Let us solve for the transfer function (or gain), G, of the simple RC low pass for the filter shown below (the output voltage V_{out} is equal to the input voltage V_{in} for low frequencies, but heads to zero at high frequencies).



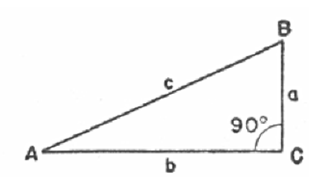
The impedance of the capacitor is frequency dependent; there is also a 90^o phase shift between the voltage across the capacitor and the current; electronic types refer to this 90^o phase shift as j. Do not worry about how we get the phase shift, assume it is valid-which it is. Anyway, the impedance of the capacitor, $Z_C = 1/j(2\pi fC)$, where f is the frequency in Hz (or cycles per second) and j is $+90^o$. The transfer function for the circuit is:

$$(V_{out}/V_{in})(f) = Z_C/(R + Z_C) = G(f) = [1/j(2\pi fC)]/[R + 1/j(2\pi fC)] = 1/[1 + j(2\pi fRC)]$$

The (f) in this equation means that G(f) is a function of frequency. We electronic engineers like to make equations as simple appearing as possible; let ω (angular frequency in radians/sec.) $\equiv 2\pi f$ and $s \equiv j\,\omega$. The transfer function may now be given as:

$$(V_{out}/V_{in})(f) = G(f) = 1/[1 + j(2\pi fRC)] = (V_{out}/V_{in})(\omega) = G(\omega) = 1/[1 + j(\omega RC)] =$$
$$(V_{out}/V_{in})(s) = G(s) = 1/[1 + sRC]$$

These equations are equivalent, only with different symbols. Looking at the denominator, remember j is just $+90^o$, we have the simple right triangle shown below;



Point B is given, in vector form, as; $B = 1 + j(2\pi fRC) = 1 + j(\omega RC)$
side b = 1, side a = $(2\pi f\,RC) = (\omega RC)$, the hypotenuse, c, is

$$c = \sqrt{[1 + (2\pi fRC)^2]} = \sqrt{[1 + (\omega RC)^2]}.$$

The phase angle, $A \equiv \theta = \tan^{-1}(2\pi fRC) = \tan^{-1}(\omega RC)$

Our transfer function is: $j2\pi f = j\omega = s$

$$G(f) = 1/[1 + j(2\pi fRC)] = G(\omega) = 1/[1 + j(\omega RC)] = G(s) = 1/[1 + sRC]$$

With a magnitude, M; $2\pi f = \omega$

$$M(f) = 1/\sqrt{[1 + (2\pi fRC)^2]} = M(\omega) = 1/\sqrt{[1 + (\omega RC)^2]}$$

and phase, θ; remember that 1/tan = -tan

$$\theta = -\tan^{-1}(2\pi fRC) = -\tan^{-1}(\omega RC)$$

Often you will see the magnitude, M expressed in dB; $M_{dB} = 20Log_{10}(M)$. Our magnitude equations, in dB, may now be given as (remember that $Log_{10}(1/x) = Log_{10}(x^{-1}) = - Log_{10}(x)$):

Magnitude, M_{dB}; $2\pi f = \omega$

$$M(f)_{dB} = -20Log_{10}\{\sqrt{[1 + (2\pi fRC)^2]}\} = M(\omega)_{dB} = -20Log_{10}\{\sqrt{[1 + (\omega RC)^2]}\}$$

Phase, θ

$$\theta = -\tan^{-1}(2\pi fRC) = -\tan^{-1}(\omega RC)$$

We are almost there. Let us see what happens when $2\pi fRC = \omega RC = 1$;

$$M(f)_{dB} = M(\omega)_{dB} = -20Log_{10}\{\sqrt{[1 + (1)^2]}\} = -20Log_{10}\sqrt{[2]} = -3dB$$

$$\theta = -\tan^{-1}(1) = -45^o$$

This is called the 3dB frequency (f_{3dB} or ω_{3dB}) and it is a constant for a given RC; $f_{3dB} = 1/(2\pi RC)$ and $\omega_{3dB}= 1/RC$. OK, RC may be given as; $RC = 1/(2\pi f_{3dB}) = 1/ \omega_{3dB}$. Substituting this into our magnitude and phase equations:

Magnitude, M_{dB}; $f_{3dB}= 1/(2\pi RC)$, $\omega_{3dB}= 1/RC$

$$M(f)_{dB} = -20Log_{10}\{\sqrt{[1 + (f/f_{3dB})^2]}\} = M(\omega)_{dB} = -20Log_{10}\{\sqrt{[1 + (\omega/\omega_{3dB})^2]}\}$$

Phase, θ

$$\theta = -\tan^{-1}(f/f_{3dB}) = -\tan^{-1}(\omega/\omega_{3dB})$$

Electronic engineers also like to use time constants, τ, where $\tau = RC$; remember $RC = 1/2\pi f_{3dB} = 1/\omega_{3dB}$, thus $\tau = 1/2\pi f_{3dB} = 1/\omega_{3dB}$. Substituting $\tau = RC$ in our magnitude and phase equations:

Magnitude, M_{dB}; $\tau = RC$

$$M(f)_{dB} = -20Log_{10}[\sqrt{[1 + (2\pi f\tau)^2]} = M(\omega)_{dB} = -20Log_{10[}\sqrt{[1 + (\omega\tau)^2]]}$$

Phase, θ

$$\theta = -\tan^{-1}(2\pi f\tau) = -\tan^{-1}(\omega\tau)$$

That's it! Below is a summary of our transfer function journey:

Transfer function; $j2\pi f = j\omega = s$

$$G(f) = 1/[1 + j(2\pi fRC)] = G(\omega) = 1/[1 + j(\omega RC)] =G(s) = 1/[1 + sRC]$$

Magnitude, M_{dB}; $2\pi f = \omega$

$$M(f)_{dB} = -20Log_{10}\{\sqrt{[1 + (2\pi fRC)^2]}\} = M(\omega)_{dB} = -20Log_{10}\{\sqrt{[1 + (\omega RC)^2]}\}$$

Phase, θ

$$\theta = -\tan^{-1}(2\pi fRC) = -\tan^{-1}(\omega RC)$$

Magnitude, M_{dB}; $f_{3dB} = 1/(2\pi RC)$ and $\omega_{3dB}= 1/RC$

$$M(f)_{dB} = -20Log_{10}\{\sqrt{[1 + (f/f_{3dB})^2]}\} = M(\omega)_{dB} = -20Log_{10}\{\sqrt{[1 + (\omega/\omega_{3dB})^2]}\}$$

Phase, θ

$$\theta = -\tan^{-1}(f/f_{3dB}) = -\tan^{-1}(\omega/\omega_{3dB})$$

Magnitude, M_{dB}; $\tau = RC$

$$M(f)_{dB} = -20Log_{10}[\sqrt{[1 + (2\pi f\tau)^2]} = M(\omega)_{dB} = -20Log_{10[}\sqrt{[1 + (\omega\tau)^2]]}$$

Phase, θ

$$\theta = -\tan^{-1}(2\pi f\tau) = -\tan^{-1}(\omega\tau)$$

NOTE

These equations can be solved using a slide rule, but it really time-consuming for the solution for a large number of frequencies. Enter the Hemmi 301 and Boonshaft & Fuchs; real time savers. The Hemmi 301 uses $\omega\tau$ equations, and the Boonshaft & Fuchs uses $u = \omega$ and $u_{dB} = \omega_{dB}$ equations.