The relationship between the transfer function $H(j\omega)$ and the various expressions for gain is somewhat obscure. For some reason, there is a reluctance to identify $H(j\omega)$ as complex number, which is a missed opportunity since as I show below, this relationship is easy to see from the polar form representation of the complex number $H(j\omega)$. And no mention is made of the fact that "j" is the imaginary unit, so that $j^2 = -1$, and that as I mention below, multiplicationn by $\pm j$ is easily interpreted and understood as a rotation by $\pm 90°$.

The transfer function $H(j\omega) = 1 + j\,(\omega/\omega_0)$ is a complex number $z = x + jy$ whose real and imaginary parts are given by

$$x \equiv \Re\{H(j\omega)\} = 1 \quad (1)$$
$$y \equiv \Im\{H(j\omega)\} = (\omega/\omega_0) \quad (2)$$

As a complex number, $H(j\omega)$ can be plotted as a point $P(x, y)$ in an Argand diagram defined by real ($\Re$) and imaginary ($\Im$) axes, thereby defining a right-angled triangle as shown in Figure 4. If the complex number $H(j\omega)$ is represented in polar form as

$$H(j\omega) = |H(j\omega)|e^{j\Theta} \equiv re^{j\Theta} \quad (3)$$

then from an inspection of Figure 4, it is easily seen that the modulus or magnitude $|H(j\omega)|$ and the argument Θ are simply given by

$$|H(j\omega)| = \sqrt{x^2 + y^2} = \sqrt{[1 + y^2]} = \sqrt{\left[1 + (\omega/\omega_0)^2\right]} = r \quad (4)$$
$$\Theta = \tan^{-1}(\omega/\omega_0) = \tan^{-1} y \quad (5)$$

When complex numbers expressed in polar form are multiplied, their arguments are added, so that multiplication by $\pm j \equiv e^{\pm j(\pi/2)}$ is equivalent to a rotation by $\pm(\pi/2) \equiv \pm 90°$.