Notes on Mechanical Fourier analyzers

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Figure 1. Jean-Baptiste-Joseph Fourier.

Introduction

Jean-Baptiste-Joseph Fourier (1768-1830) for a mathematician had a very adventurous life [5].

- Not being of noble birth, he was not eligible to attend the military college.
- However, he was whisked out of his job at the *École Polytechnique* and sent in 1798 to Egypt to work on the Commission of Arts under Bonaparte's debacle.
- After the above fiasco and returning to France, Bonaparte sent him as *Préfet* to the French province of Isére.
- He became interested in the theory of heat transfer. This is outside of the scope of this paper, but apparently involved the solution of partial differential equations who solutions are determined by boundary value conditions.

$$
\begin{aligned}
X(f_c) &= \int_0^T x(t)e^{-j2\pi f_c t}\, dt \\
&= \int_0^T x(t)[\cos(2\pi f_c t) - j\sin(2\pi f_c t)]dt \\
&= \int_0^T [x(t)\cos(2\pi f_c t)]\, dt - j\int_0^T [x(t)\sin(2\pi f_c t)]dt
\end{aligned}
$$

Figure 2. The form of the Fourier transform of interest.

The form given here with t being the independent variable, is a common usage in engineering. Note that the original integration breaks down into two real integrations:

- $x(t)$ is multiplied by either the sine or cosine of $2\pi f_c t$, f_c being the frequency of interest.
- The product is then integrated over the finite interval $[0,\ T]$.
- In order to implement this operation, we must be able to mechanically generate the sine waves, multiply them times $x(t)$ and then mechanically integrate the resulting products over the interval $[0,\ T]$.

Applications

The applications of the Fourier transform are so pervasive that its implementation will be found in many fields of science and engineering. Here, to mention a few, are some of them:

- Tidal analysis and prediction seems to have been the first.
- Analysis of sound in general, and music and musical instruments in particular.
- Analysis of electrical circuits.
- Analysis of mechanical systems.
- Astronomy.
- Probability and statistics. The covariance (correlation) function is the Fourier transform of the probability density function.
- Many digital cameras use the Fourier transform as part of a data compression algorithm.
- Computer Aided Tomography (CAT) scans, widely employed in medical analysis.

Lord Kelvin's Tide Prediction System[1]

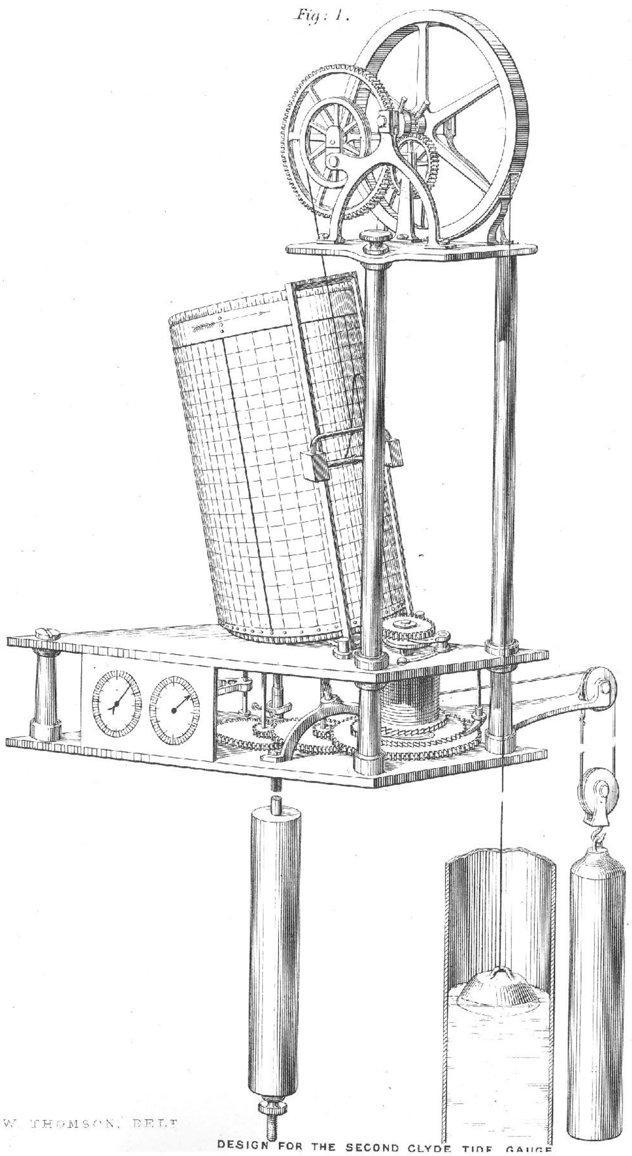

Figure 3. The tidal gauge.

The system [18] is used regularly to collect tide information at a given location, analyze the amplitude and phase of a set of known frequencies to be found in the tide and then, based on the collected information to predict the tide at that point in future times.
The system consists of three parts:

1. Shown in Figure 3 is the tidal gauge. It consists of a float sitting in the water, a drum that records the height of the float relative to the drum, a pen marking the tide level on the drum, and a clockwork mechanism that turns the drum once in 24 hours.

2. Shown in Figure 4a is a drawing of the Kelvin harmonic analyzer. Note this is a special device designed to work at the frequencies to be found in the tidal action as caused by the periodic motions of the earth, moon, sun, etc. Figure 4b is a photographic detail of the machine.

3. Figure 5 is the tide prediction machine, which employs the information gathered by the tide gauge and produced by the analyzer to extrapolate what the tide will be at a given time in the future.

The harmonic analyzer employs eleven *ball and disk integrators* in its computations. This device was invented by Lord Kelvin's brother. James Thomson [18]. The integrator evaluates the integral $y = \int_0^T h(t)x(t)dt$ (where $h(t)$ is the water heighth, $x(t)$ is either the sine or cosine as a function of time for a given frequency, and T is length of time over which the integral is computed in the following manner: the integrator consists of a disk, set at about 45° in this implementation, whose angular rotational position is set to correspond to $x(t)$. Lying on the disk is a heavy metal ball whose horizontal position from the center of the disk is proportional to the $h(t)$ term. The ball in turn is also resting on a metal cylinder, also horizontal to the disk and of the same width. As the disk turns, motion is imparted to the cylinder by way of the ball. The cylinder's rotational position is then proportional to the integral of the $h(t)x(t)$ term.

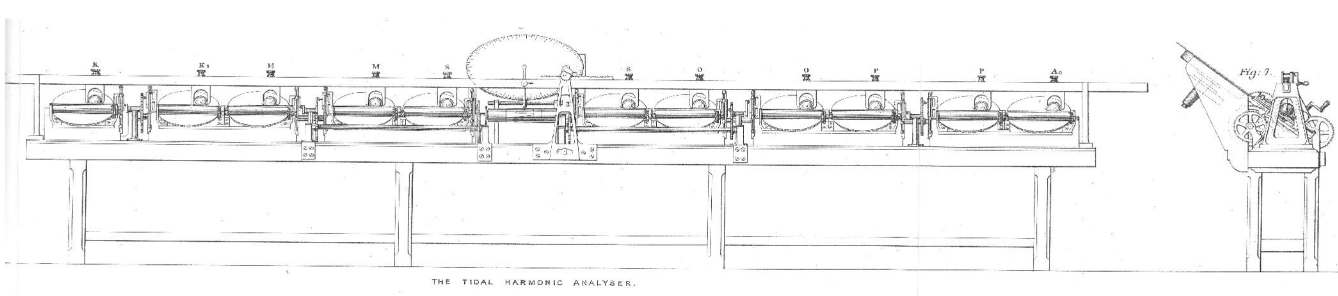

Figure 4a. The Harmonic Analyzer.

[1]William Thomson will be referred to only as Lord Kelvin in the following discussion.

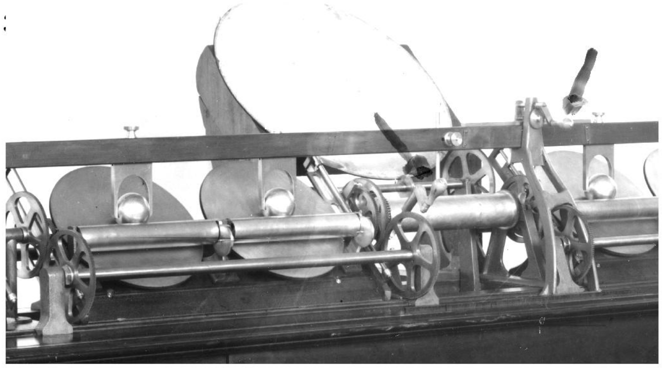

Figure 4b. Detail of the Tide Analyzer.

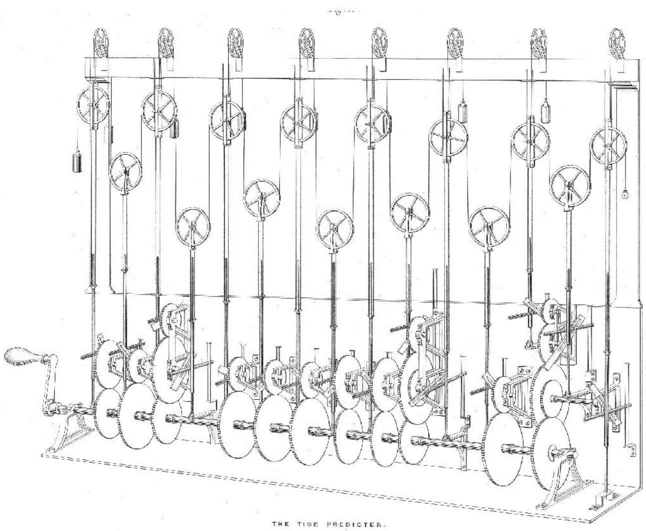

Figure 5. The Tide Predictor.

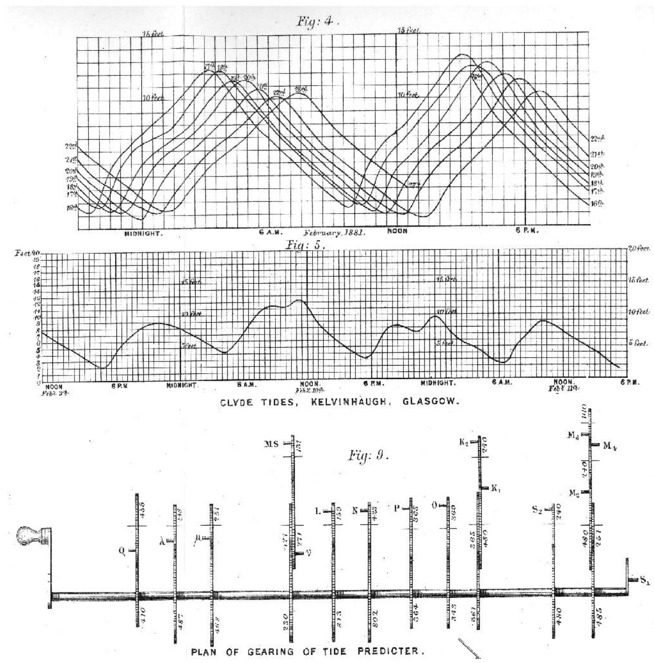

Figure 6. Details and results from the Tide Predictor.

In Figure 4b note the two arrows pointing at the two control cranks of the analyzer. The one on the right moves the long horizontal bar back and forth as appropriate. There is an indicator on this bar that nearly touches the cylinder directly beneath the large circular plate. The indicator is used to follow the tidal trace on the recording paper as it passes over the cylinder. Additionally, this crank causes the large bar that runs the length of the instrument to move. The balls of all the ball and disk integrators are held by loose clamps on this bar, so that bar movement left or right causes all of the balls on all the integrators to move the same distance.

The crank on the left when turned causes two movements: the paper roll with the readings from a tidal gauge to be fed slowly through the machine, and the harmonic frequencies to be applied to the disks, causing them to rotate at the correct four frequencies. Note that each pair of integrators have a single frequency, but they are implemented such that are out of phase by 90°.

Remembering that “the tide ebbs and flows twice in twenty-four hours”, the “fundamental” angular frequency per hour is approximately $360°/12 \approx 30°$ per hour. Prior work by others [18] indicated that there were four frequency components that needed to be fitted. Without going into details, I cite from [18], page 12, the following table:

The numbers shown below I believe to be the actual computed gear ratios that would yield the nearly correct frequencies to be employed both in the analysis and in the predictions.

Speeds of the several Tidal Constituents in Degrees per Hour.

—	Numerical Approximation.	True Speeds.	Differences.
M	$30° \times \frac{184 \times 256}{199 \times 245} = 28°{\cdot}9841042$	28°·9841042	0·0000000
K_1	$30° \times \frac{119 \times 317}{209 \times 360} = 15°{\cdot}0410686$	15°·0410686	0·0000000
O	$30° \times \frac{58 \times 92}{89 \times 129} = 13°{\cdot}9430363$	13°·9430356	0·0000007
P	$30° \times \frac{178 \times 221}{242 \times 326} = 14°{\cdot}9589312$	14°·9589314	0·0000002

Each integrator records the angular movement. At the end of the reading session, the total movement read from each pair of integrators for a single component can be converted into the desired amplitudes (sine and cosine) of that tidal component.

The Tide Predictor

The third device in the system, the Tide Predictor, is shown schematically in Figure 5. From [18], page 17:

> “The object is to predict the tides for any port for which tidal constituents have been found by the harmonic analysis from tide-gauge observations: not merely to predict the times and heights of high water, but the depth

of water at any and every instant, showing it by a continuous curve, for a year, or for any number of years in advance."

This device computes the equation:

$$H(t) = a_0 + \sum_{k=1}^{K} [a_k \cos \omega_k t + b_k \sin \omega_k t]$$

where a_0, a_k and b_k, $(k = 1, K)$ have been evaluated by the analyzer.

The Tide Predictor works in the following manner:

- There is a flexible wire that runs over the higher pulley wheels and under the lower pulley wheels, as shown in Figure 5. Note that the mechanism in the figure does not exactly agree with the description.
- The upper pulleys are counter weighted.
- Driven by a crank, the mechanism at the bottom generate the sines and cosines with appropriate frequencies and amplitudes, one trigonometric function per lower pulley.
- These forces affect *all* of the pulleys. That is, all of the pulleys can move up and down as pushed by the rods attached to the mechanism.
- The machine adds all of the displacement terms with the resulting tidal value appearing as the position of the weight on the left.

The machine was very successful and was adopted by many countries. The United States [15] was using the Tide Predictor well into the 20th century[2]. In the US usage, the tide predictor was employed, but with a different method used to evaluate the coefficients.

Lord Kelvin notes that four hours of crank turning would produce tidal predictions for one harbor for a full year.

The Henrici Harmonic Analyzer

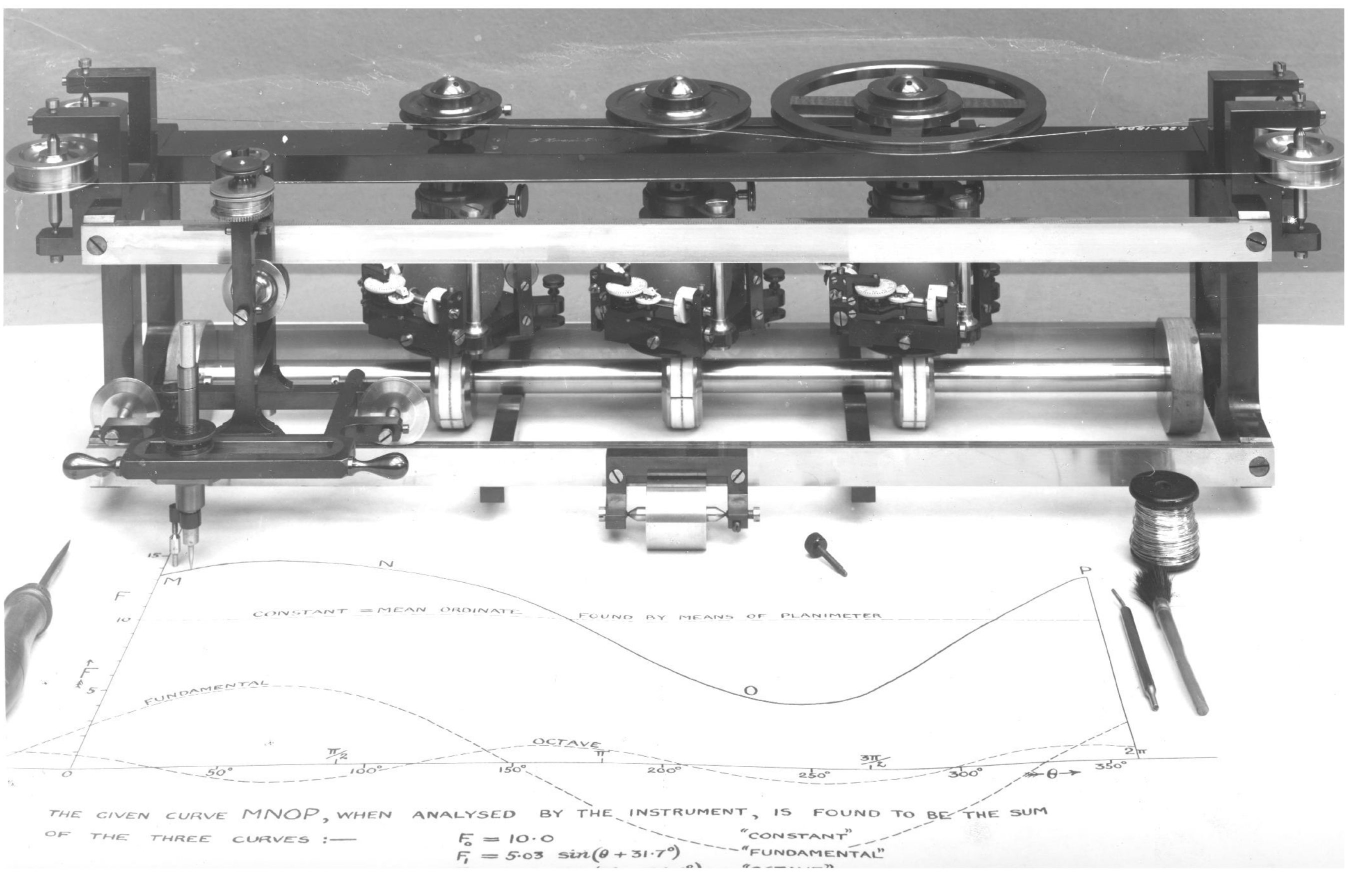

Figure 6. The Henrici Analyzer. Photograph courtesy of the Science Museum, London.

The Harmonic Analyzer of O. Henrici.
The Henrici analyzer [7, 8] is a much more compact machine than that of Lord Kelvin, noting that the device, at least to me, turns out to be somewhat larger than I guessed it to be from the above photograph when I finally saw it at the Science Museum.

The machine is such that the x axis runs from left to

[2]A student at a seminar told me this in the 1970s and also mentioned this reference.

right and the y axis is forward. I suspect that depending on how the machine is set up, the direction of the y axis could be either forward or backward.

The analyzer is not of the ball and disk variety, but it does employ *glass* balls for a similar purpose: they are difficult to see in the picture. There are three of them in this version of the machine, and they are under the wheels at the top of the machine in a metal cage (or socket) that holds them. Each such ball has two readouts attached to its cage with their inputs touching the ball. They are spaced at 90°. These readouts are similar to those found on an ordinary planimeter. Also, the bottoms of these cages are open and each of the balls is resting on a single roller which runs the width of the machine.

Thus, when the machine moves (the whole machine can move only in the y direction), the balls roll in their sockets causing the readouts to change values.

At the far right end of the picture will be seen a spool of wire. This is not an accident, for wire plays an important part in the operation of the machine: note that on the top of the machine there is a single such wire that wraps around each of the three wheels and continues on. There is a small carriage on the left end of the machine, directly in front. The wire, after going totally around each of the three wheels and then around a fixed pulley at each end, finally attaches to the top of this carriage.

The whole machine employing movement in y direction only and the carriage providing movement in the x direction only, is used to trace out the the curve to be analyzed. Any *carriage* movement in the x direction will cause the wire to turn the wheels on the top of the machine and thus impart a circular motion to the cages containing the balls. That is, this is the way that the sinusoidal is input to the integration.

If you scale off the diameter of the large wheel on the right and then multiply this value by π, the resulting distance corresponds both to the length of the fundamental frequency and the working width of the machine. Thus, this wheel determines how long the cycle of the lowest frequency can be. In any event, the amount that the small carriage can move in the x direction fixes the maximum value of the x input, so that the fundamental should be less than or equal to this.

Seashore [16] employed the Henrici in analyzing the timbre of musical instruments. He had a musician play the notes of an instrument one by one, recording a single cycle of a note, I believe, on smoked glass. This note would be scaled to fit the fixed input length of the analyzer, and the three harmonics, phase and amplitude, would be computed. For the next set of three harmonica, the machine would have to changed: the wire taken off, different wheels put on, and new wire added before the next set of measurements could be taken. Examining a single instrument would be a time-consuming chore[3].

It is worth mentioning that using the Henrici analyzer, Seashore found that one note on the bassoon has no fundamental. Your ear (including your brain) just thinks that it hears one.

The Stanley Harmonic analyzer—1947

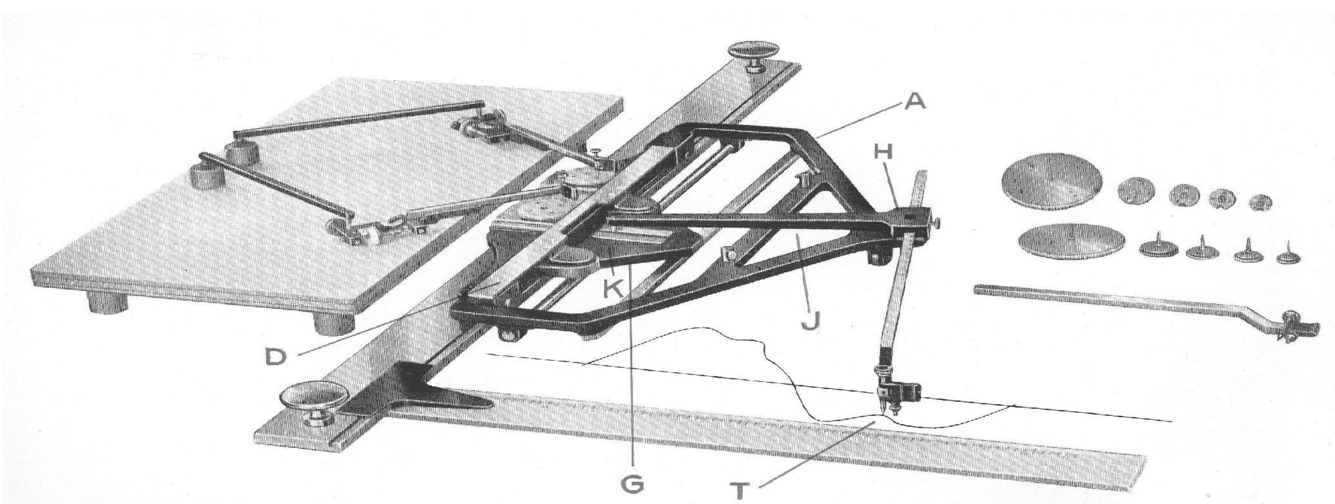

Figure 7a. The analyzer as shown in the instruction booklet. It is the five-harmonics version.

[3]But that is what graduate students are for.

Figure 7b. The Yule, Mader, Ott analyzer, as made by W.F. Stanley.[4] It processes 18 harmonics.

An analyzer Using Planimeters

The Kelvin Tide analyzer and the Henrici Analyzer are both different forms of ball type integrators. And both machines can process more than one complex sinusoidal term at a time.

The Henrici has some disadvantages: it is limited in the x direction span by the diameter of its primary wheel. And while it can potentially handle many frequencies at one time, if there are more frequencies to process than a particular implementation can handle, switching to another set would seem to be clumsy, involving “rewiring” the device, so to speak. Additionally, the device appears to be expensive, though I have no pricing information to validate this claim.

G.U. Yule [19] in 1895 proposed a more “economical” version of the Fourier analyzer employing ordinary planimeters. The implementation of this idea seems to have been done by O. Mader, who published a paper on the subject in 1909 [12][5].

The original machines seem to have been done by Gebrüder Stärzl at München in 1909. Subsequently it was made by Ott in Germany. Finally, the machine at hand was manufactured by W.F. Stanley & Co. in England.

This device is limited in that it can only process one frequency at a time. However, changing to another frequency appears to be simple relative to that required for the Henrici. In addition, changing the length of the fundamental is, within a sizable range, relatively easy to do, as opposed to the fact that the Henrici has a fixed fundamental period.

As shown in Figure 7b, the analyzer is stored in a box not quite a meter in length; the complete set, including the box, weighs about 20 kilos. Thus it is transportable

[4] Author’s collection.

[5] I do not know if Mader published an earlier paper.

as opposed to being truly portable. The two small boxes on the right each contain an Albrit planimeter (made by Stanley). They are quite ordinary.

The large white square slightly left of center is the table for the planimeters as shown in Figure 7a. At the back of the box is a long metal track, almost one meter in length, that supports the triangular carriage shown in Figure 7c. At the left of the box is the combination of the two carriages that are used; these are shown in more detail in Figure 7c.

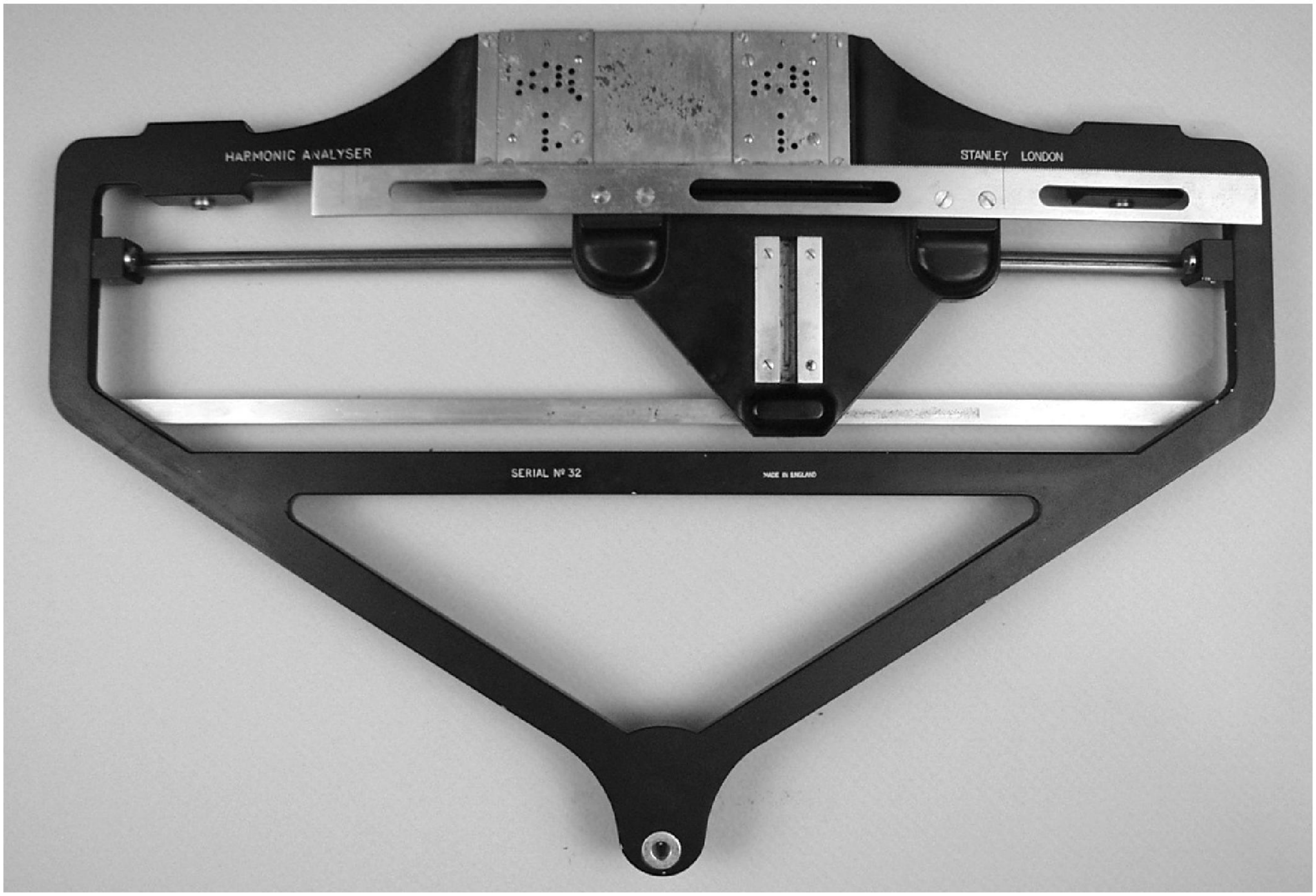

Figure 7c. The main carriage with the smaller, secondary carriage on top of it.

The main carriage has two beveled wheels underneath it that run in the groove in the separate, long metal track. At the top of the carriage as shown, there are two sets of 18 holes each. Each pair of gears (18 pairs in total) for a frequency has pins on their back side, and these pins fit into one specific matched pair of holes. The gears themselves engage into the long rack on the chrome plated bar *which is attached only to the smaller carriage.* The smaller carriage runs on a round bar with a slot on top within the main carriage. Both carriages travel in the same y direction.

The tracing pins of the planimeters fit into one of two holes in each pair of gears. Movement of the small carriage causes the gears to rotate, thus inducing a sinusoidal input into the two planimeters. On the other hand, if the small carriage is fixed and the large is moved, there is no input to the planimeters.

The machine has two tracing bars, one of which is chosen for taking readings. The one selected is input to a square hole in the side of the (black) bar H. The black bar in turn has a pin at the bottom which fits into the socket under H on the main carriage as shown in Figure 7c. The other end of the black bar has a pin which fits into the slot in the small carriage.

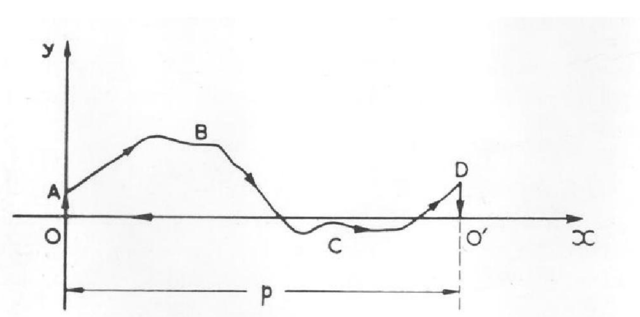

Figure 7d. An example of a curve to be processed.

The first step in processing the curve in Figure 7d would be to determine the area under the curve following the route OABCDO'O. This could be done using any planimeter, perhaps one of the two in the box. The position of the tracing arm would be adjusted to fit the length of the data segment.

After that, the values of the coefficients of the frequency terms desired (no more than 18) would be found one frequency at a time. To do this, the pair of gears for the frequency would be selected and inserted into their proper holes and oriented in the proper direction. The two planimeters would then be set up as shown in Figure 7b. The tracing arm would be set to point O and the readings on the two planimeters noted.

Then the tracing arm would be moved around the path OABCDO'O as before, and the current planimeter readings would be taken. The starting planimeter values for the frequency would be subtracted from the final one, and the results scaled to whatever units are desired.

This procedure would then be repeated for the other harmonics.

Notes and Conclusions

The advent of digital electronic computers eliminated the necessity for these specialized mechanical analyzers. Even without the popularization of the the Fast Fourier Transform (FFT) in 1964, the solution to the tidal problem would have been easy to implement on any garden variety computer after 1955 as there are only eight sinusoidal terms, and they could be generated recursively. Also, the Blackman-Tukey procedure for estimating power spectral densities (PSDs) had been available from the 1950s.

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