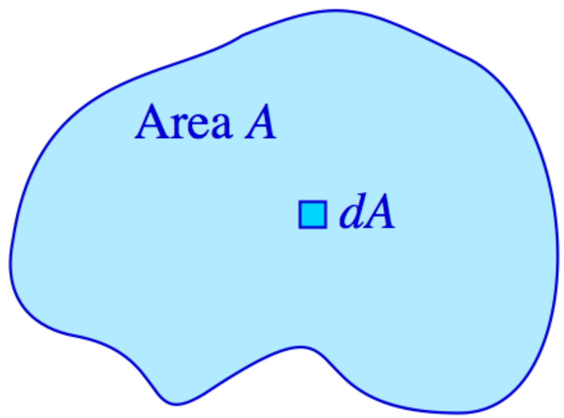
Planimeters

The mechanical measurement of Areas, Moments of Inertia, and the like.

Planimeter, *n.*: An instrument with a movable pointer used for mechanical measurement of the area of an (irregular) plane figure. (oed.com, 2013)

Integrator, *n*: One who or that which integrates; *spec*. an instrument for indicating or registering the total amount or mean value of some physical quantity, as area, temperature, etc.

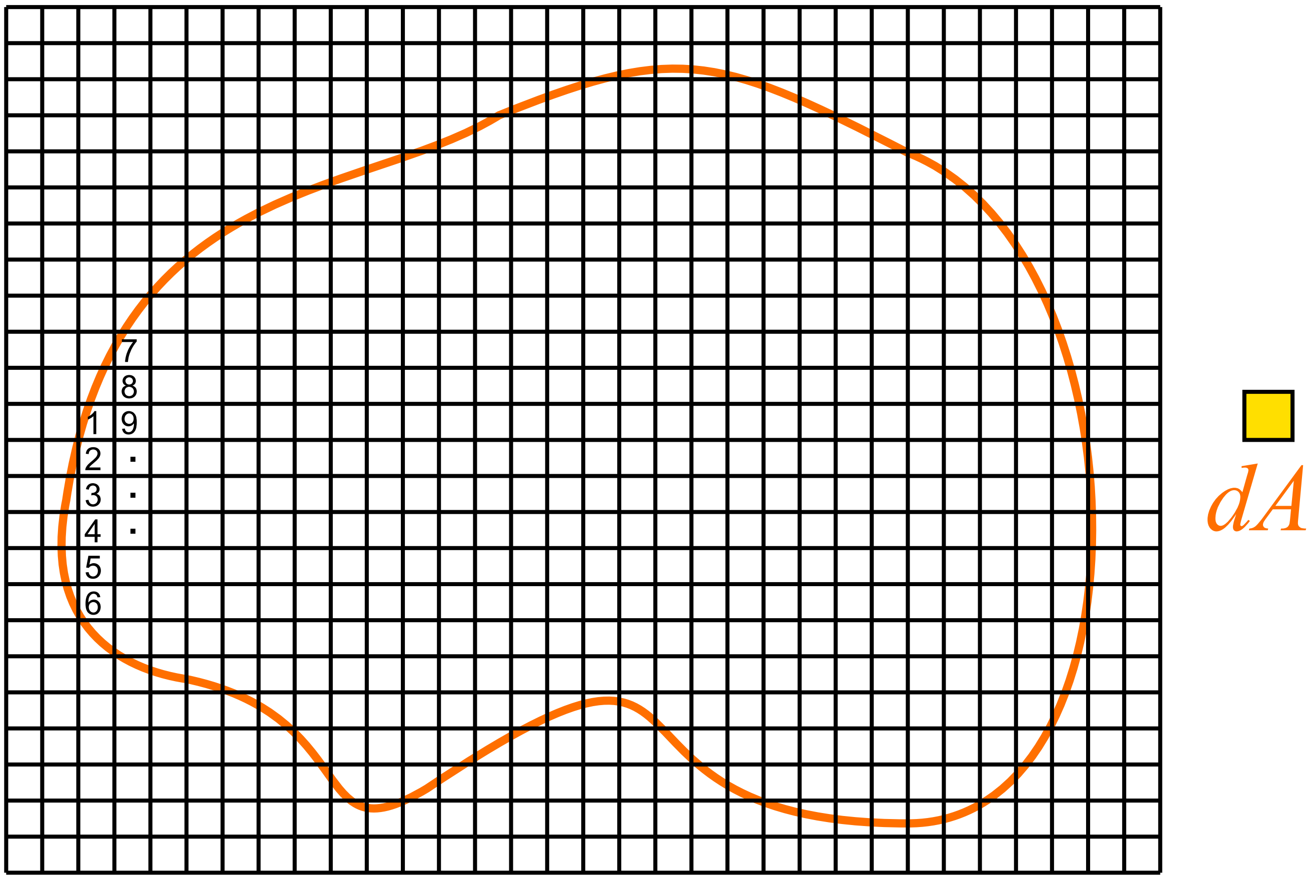


$$A = \int_A dA$$

Several mechanical gadgets were invented in the 1800's.
The most successful was invented by Swiss mathematician Jakob Amsler in 1854.
He founded a company to manufacture the planimeter, and then invented gadgets to determine more complicated integrals.

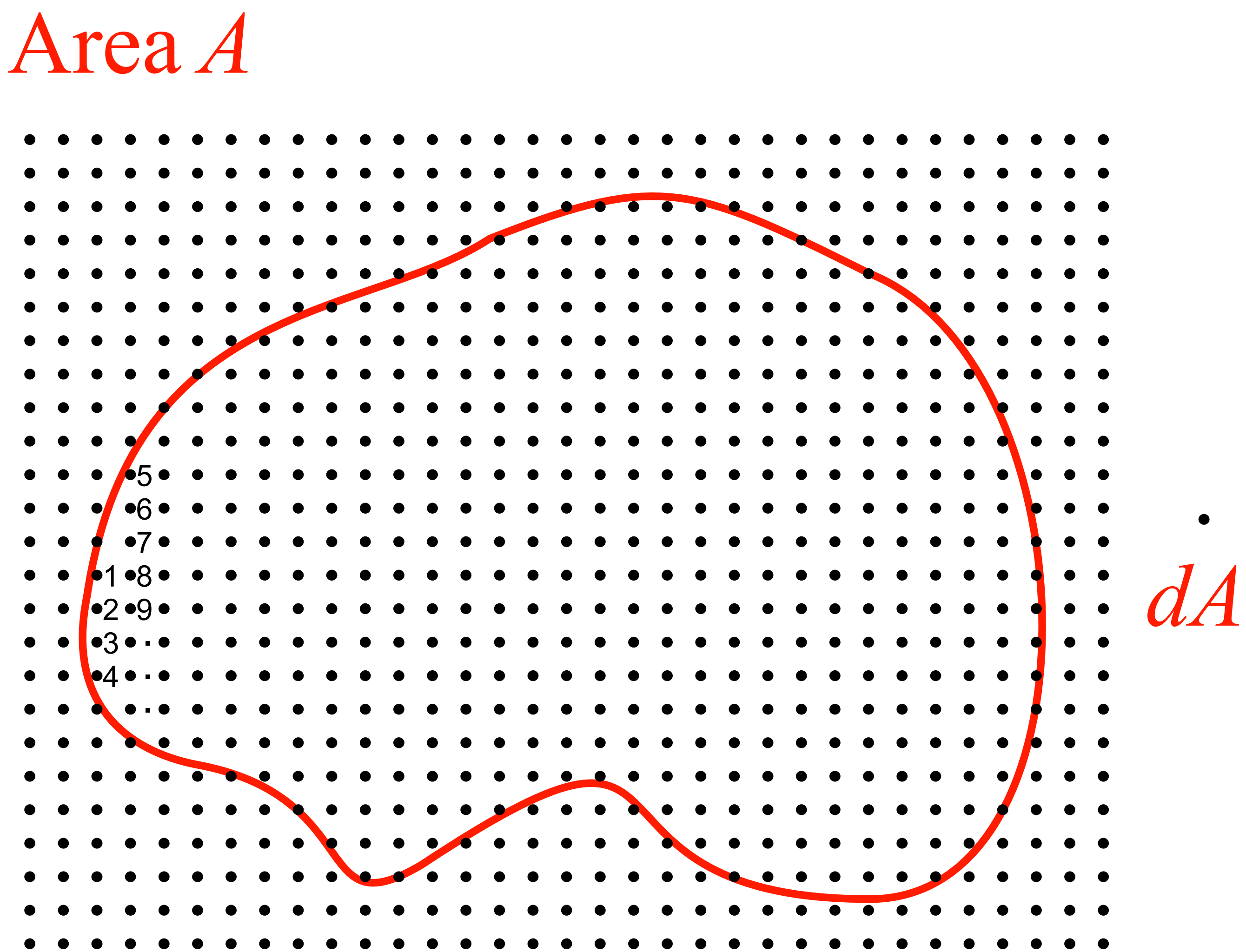
Brute Force Measurement of Area

Counting Squares:

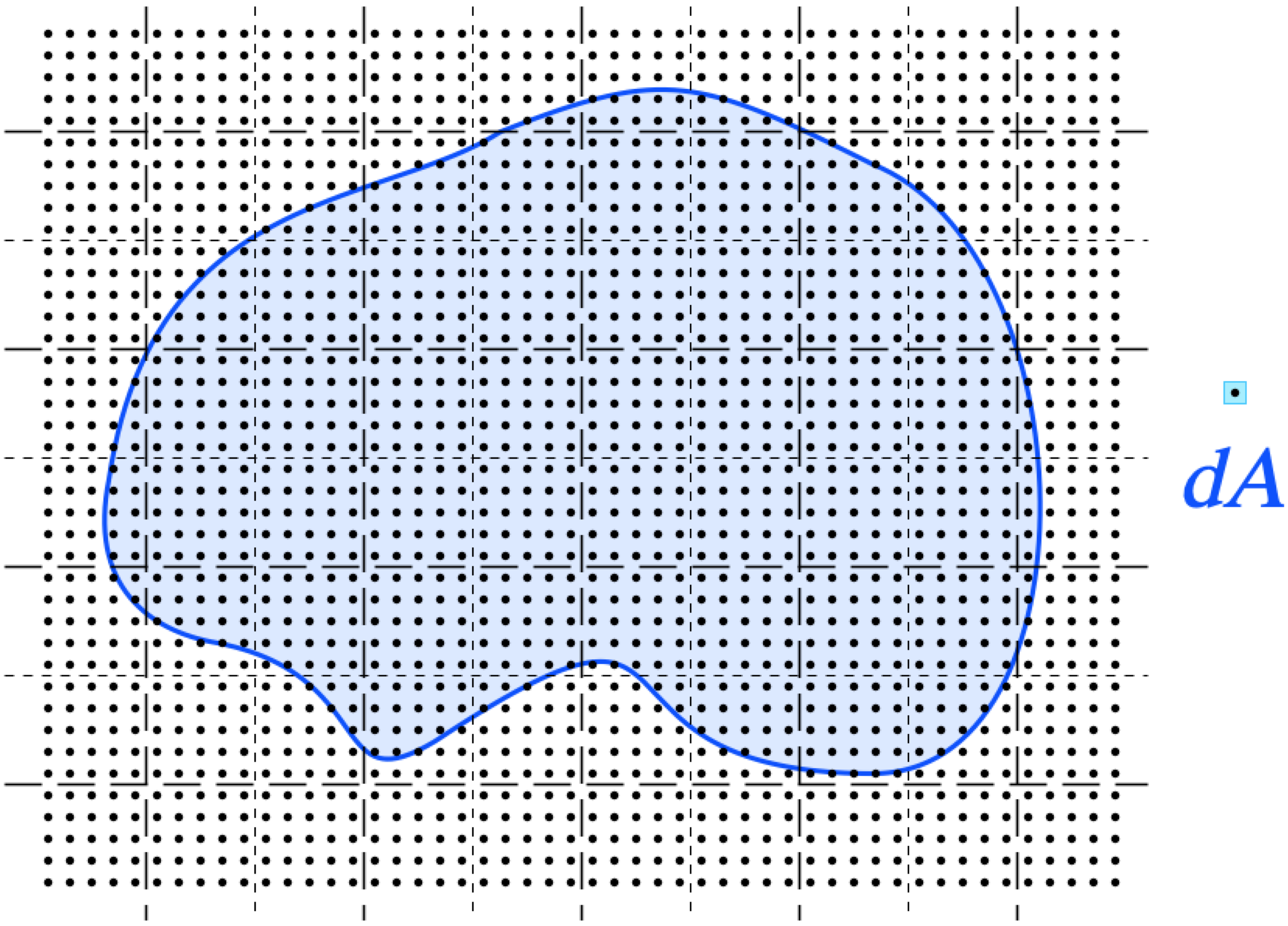


The more laborious, the more accurate.

More conveniently, Counting Dots

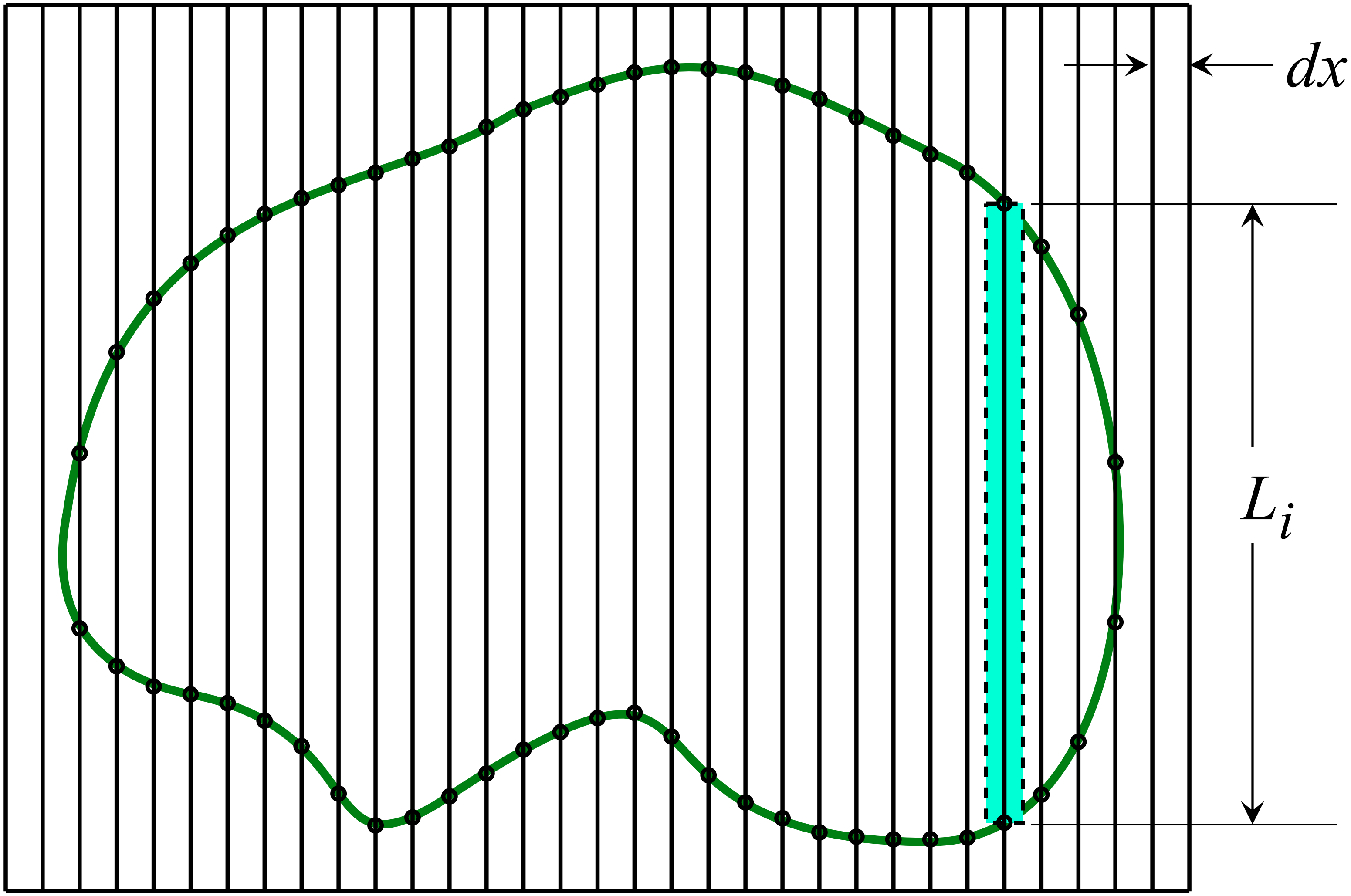


Even more conveniently, dots with a coarser grid.



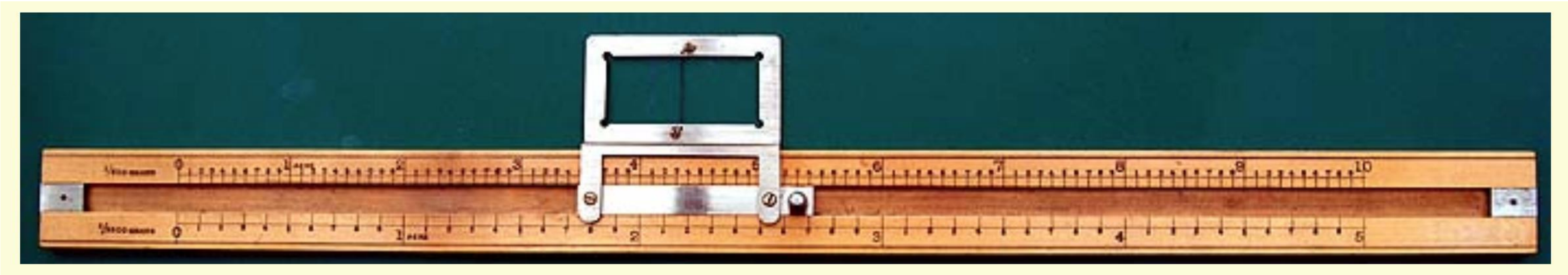
A few different papers in the early 20th Century have presented the idea of counting dots on a transparent overlay.

Area by Measuring Line Segments

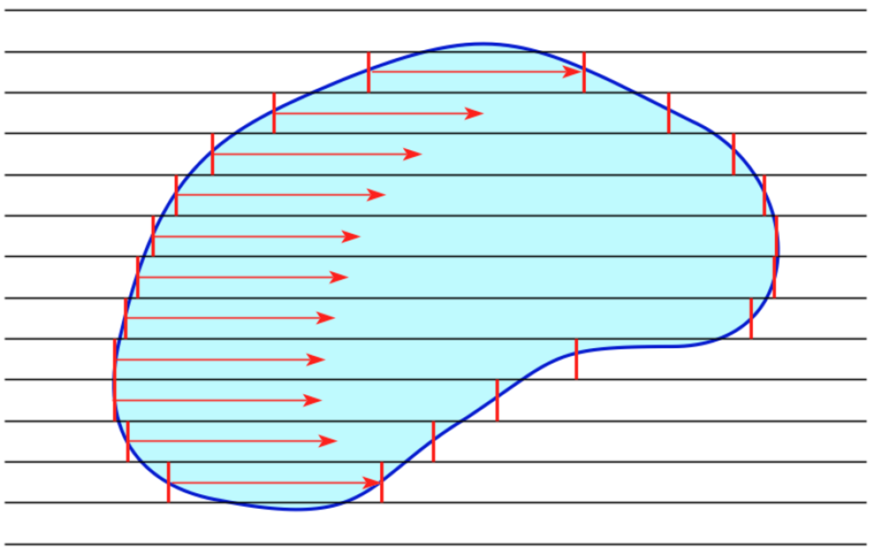


$$A = \square\, L_i dx$$
$$i$$

Linear Scale for Area Measurement



(Cooke, Troughton & Simms, area measuring scale)



A grid of straight lines on a transparent (*e.g*. celluloid) sheet is placed over the area to be measured. The scale is then placed successively along each of the lines. For each line the hairline marker is moved across the area to add successive increments of length into a running sum.

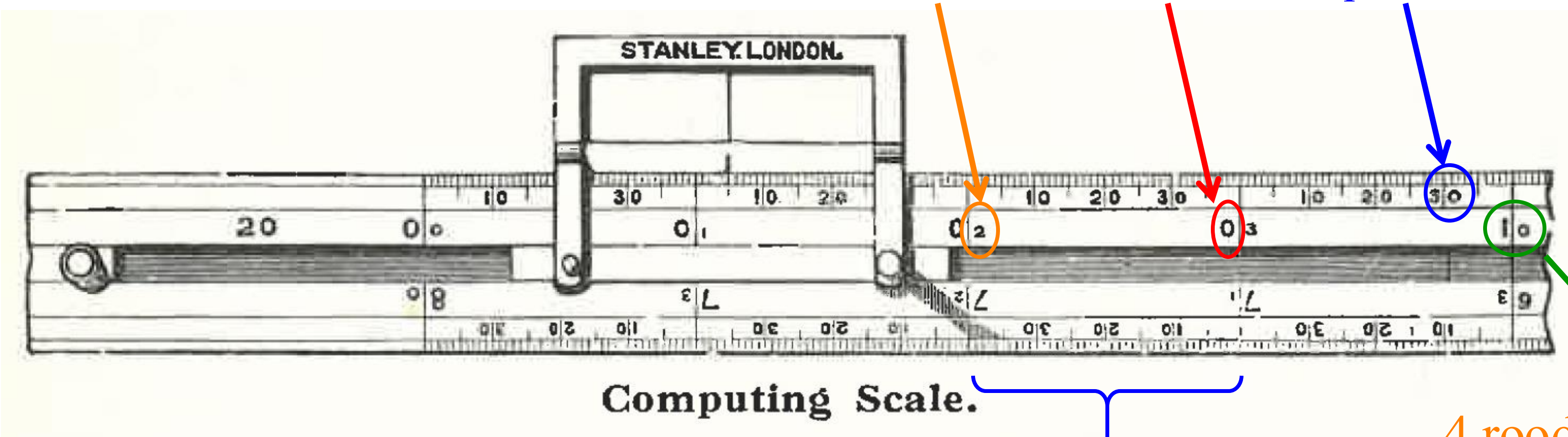
(photo: www.mathsinstruments.me.uk/page 34.html)

Scale for Measuring Area in Surveyor's Units

Designed for units of rods, roods and acres.

1 rod = 16.5 feet

roods   # acres   # square rods



4 roods = 1 acre

40 square rods = 1 rood

This ruler with its fixed scale is designed for one particular scale of map.

(sketch from W. F. Stanley, Mathematical Drawing Instruments, *e.g.* 8th ed. 1925, pp. 247-251)

Hair Planimeter designed by A. Beuviére

Figure is placed under the scribed glass slide;
Carriage with measuring disc is moved repeated over the figure.

https://irem.univ-reunion.fr/calculsavant/Exposition/ArtsetMetiers/06710-0000-e.html

Hair Planimeter

Another view.

system Beuvière, 1844. Maker Niggl, Paris, Inv. 06245-0000.
Musée des Arts et Métiers - CNAM, Paris, photo M. Favareille

Modern Measurement of Area
(by software)

App Store Preview

This app is available only on the App Store for iPhone and iPad.

Planimeter — Measure Land Area 4+

Measure map area and distance

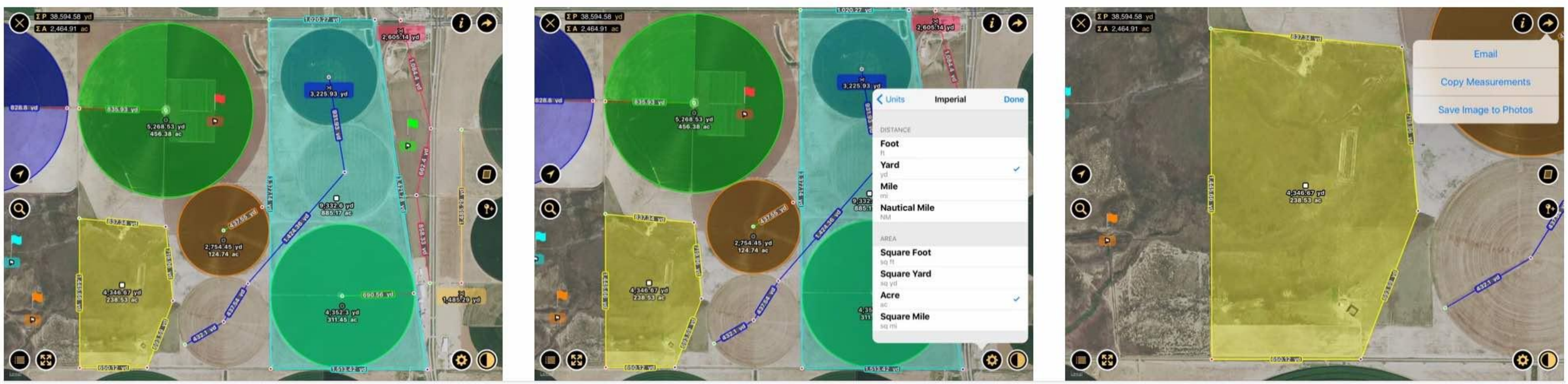
Core Signals

#64 in Productivity

★★★★★ 4.7 • 1.5K Ratings

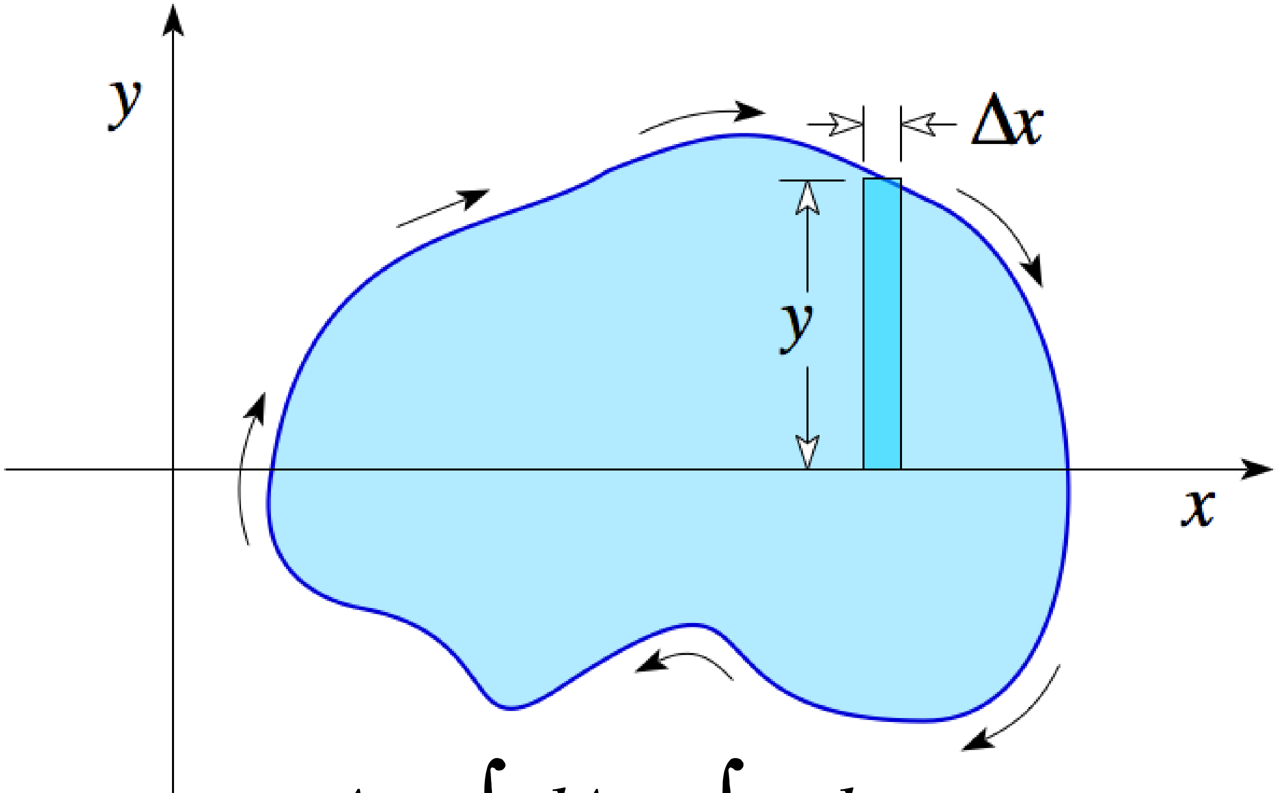
$7.99

Screenshots iPhone iPad



https://apps.apple.com/us/app/planimeter-measure-land-area/id423492040#?platform=ipad
DL 5-23-2021

Area as the integral of rectangular increments

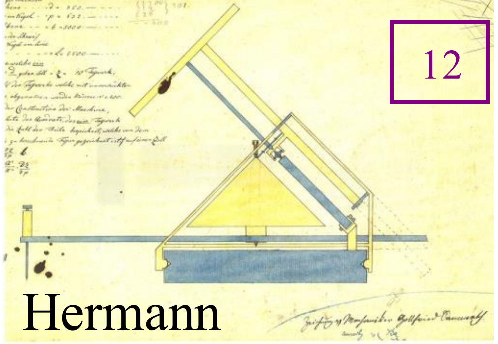


$$A = \oint dA = \oint y\,dx$$

Recorded increments of area must have magnitude proportional to *y* as we scan over *x*. Early designs recorded the integral on a measuring dial on which an indicator pointer moved according to *y* times increments of *x*. The idea was to make a linkage between the *y* value and the rate of dial rotation. Early designs used a cone or a disk for this purpose.

Several Inventors Tackled the Problem

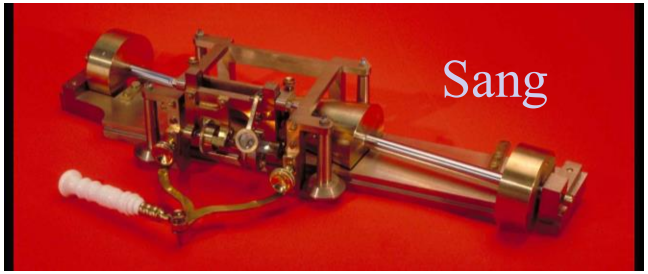
- Johann Martin Hermann (1814)
  Bavarian Land Surveyor–never published–created for his own use.
  No example survives, but we have his design figure.



* Tito Gonnella (1824)
  Prof. at Univ. of Florence. Published – did not commercialize.

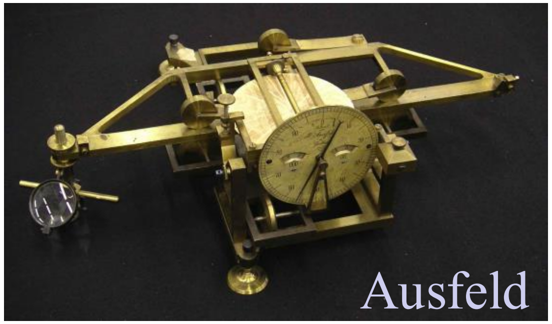
- Johannes Oppikofer (1827)
  Swiss Inventor. Mfg. in France by Ernst (1836+); then Claire.

* Kaspar Wetli (1848)
  Swiss Engineer – Austrian patent - replaced cone with disc.
  Mfg. by Georg Christoph Starke of Vienna.



* John Sang (1851)
  Scottish Inventor. Cone based rolling “platometer” .

* Hermann Ausfeld (1851)

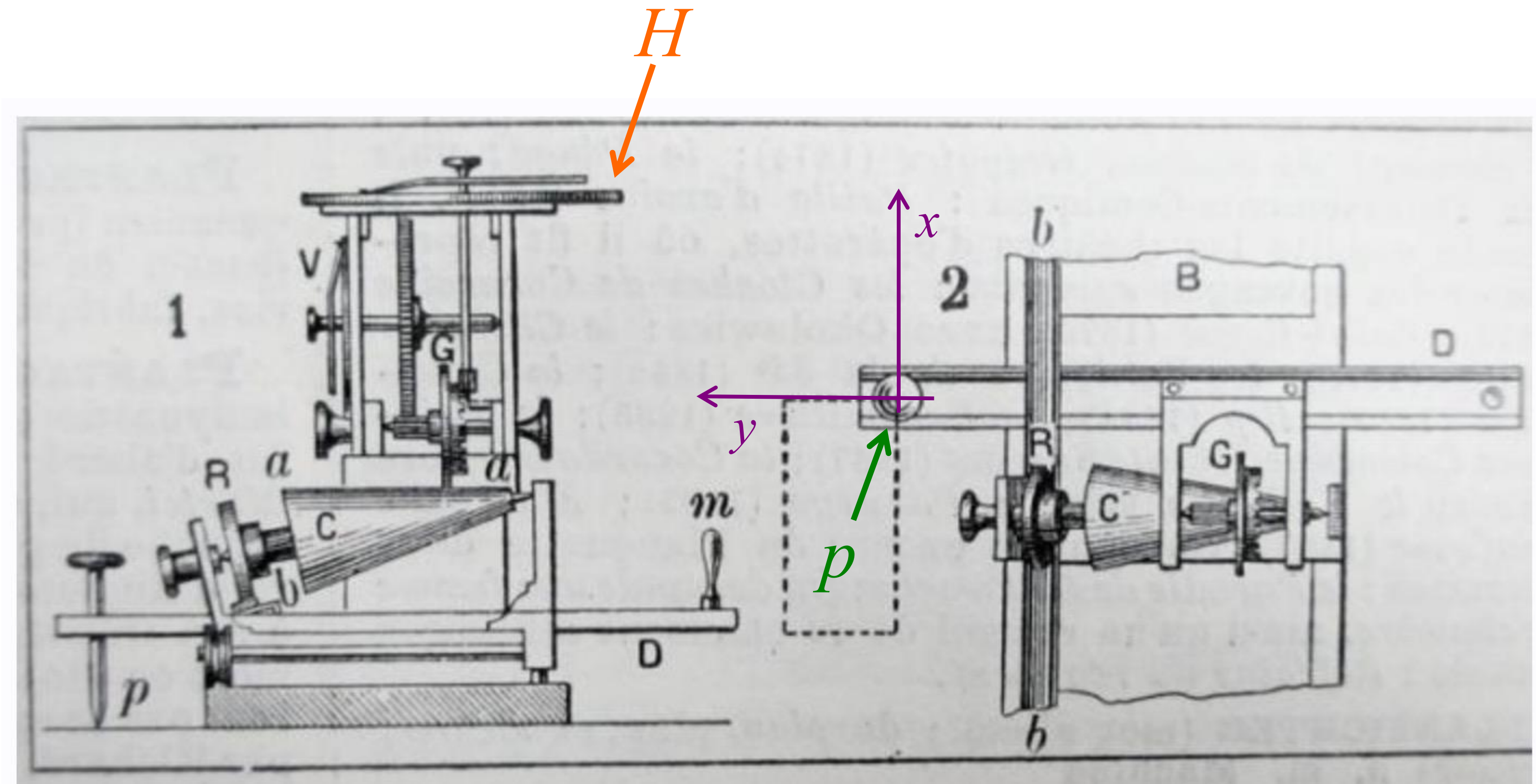


- Jakob Amsler (1854)
  Swiss Professor of Mathematics. Founded Amsler & Son.
- James Clerk Maxwell (1855)
  Scottish Mathematical Physicist.

*Displayed at the Crystal Palace Exhibition, Kensington, London,1851.

Earliest Planimeters were Complicated Gadgets

Cone planimeter by Johannes Oppikofer, Germany, and Ernst, Paris, 1827 - 1835

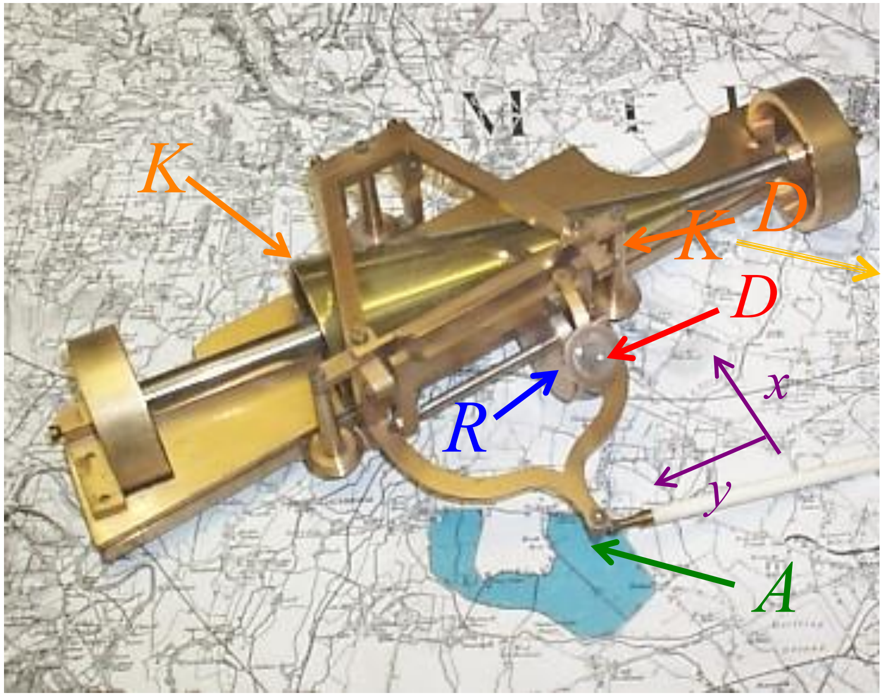


Planimètre Oppikofer et Ernst : 1. Elévation ; 2. Plan.

Position of wheel *G* on cone *C* depends on *y* value as given by position of pointer *p*.
Dial *H* records the rotation of wheel *G*.

Sang Planimeter of 1851

Another complicated Cone-Based design

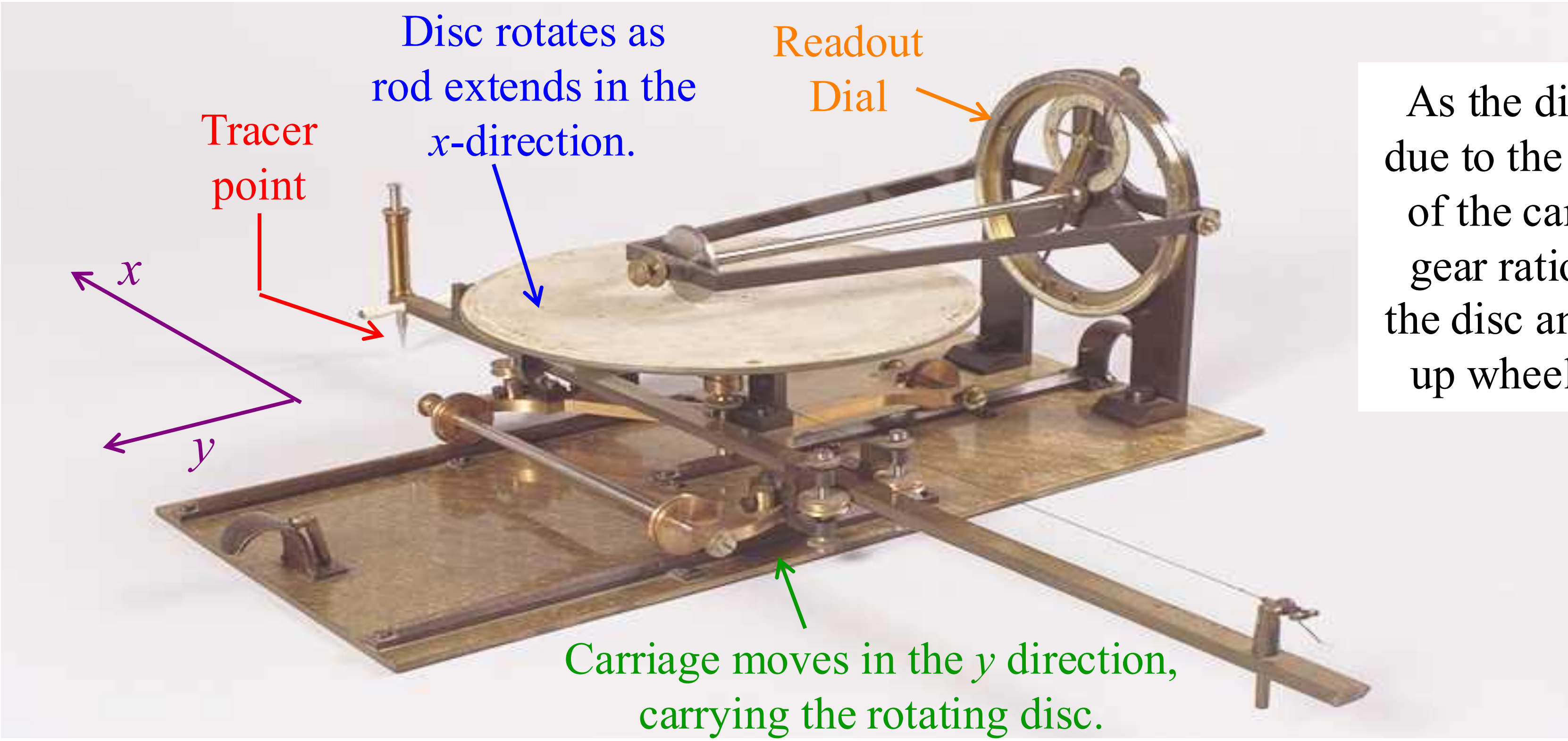


Position of wheel *R* on cone *K* depends on *y* value as given by position of pointer *A*.
Dial *D* records the rotation of wheel *R*.

(Source of photograph uncertain.)

Wheel on Disk Planimeter

Wetli-Starke Planimeter, patented 1849.



As the disc moves, due to the ± *y* motion of the carriage, the gear ratio between the disc and the take-up wheel changes.

Photograph from Technik Museums der Technischen Universität Delft.
http://www.history.didaktik.mathematik.uni-wuerzburg.de/ausstell/planimet/starke.html (2017)

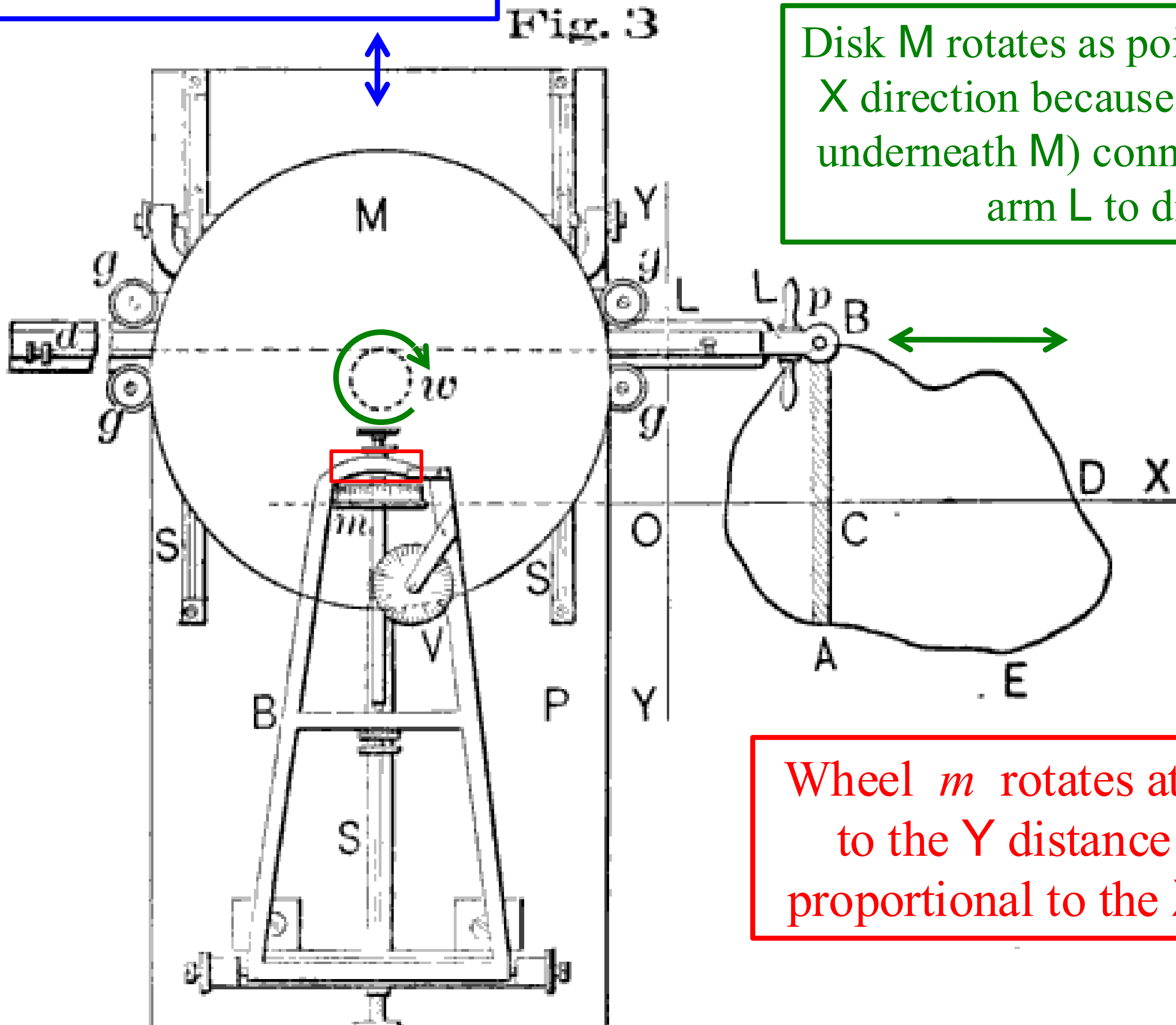
See article by Charles Care, A Chronology of Analog Computing, The Rutherford Journal – The New Zealand Journal for the History of Science and Technology

Early Planimeters

Wetli-Starke Planimeter, patented 1849.

Disk M moves along track S as the pointer B moves in the Y direction.

Disk M rotates as pointer B moves in X direction because gear *w* (hidden underneath M) connects the pointer arm L to disk M.

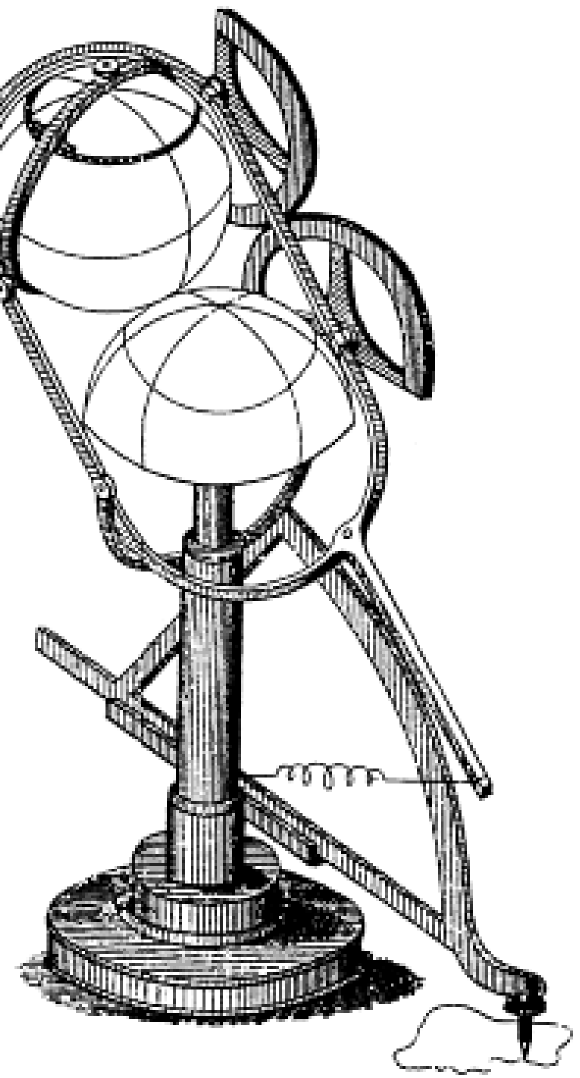


Wheel *m* rotates at a rate proportional to the Y distance of pointer B and proportional to the X speed of point B.

Mechanical Integrators including the various forms of Planimeters, by Prof. Henry S. H. Shaw, D. Van Nostrand, 1886.

Another Early Planimeter

Maxwell's Planimeter, 1855

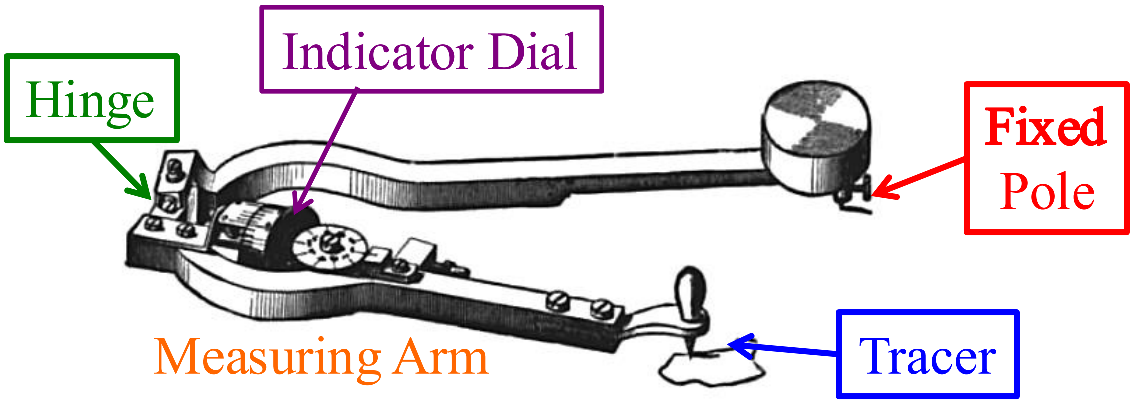


Invented by James Clerk Maxwell in 1855 while he was at Trinity College, Cambridge. (The Maxwell of Maxwell's Equations of electromagnetism, etc., etc.)

Maxwell had not yet heard of Amsler's invention, and abandoned the project after learning about Amsler's simpler and more practical device.

from website of Prof. Robert L. Foote, Wabash College, Crawfordsville, Indiana
http://persweb.wabash.edu/facstaff/footer/Planimeter/PreAmsler.htm

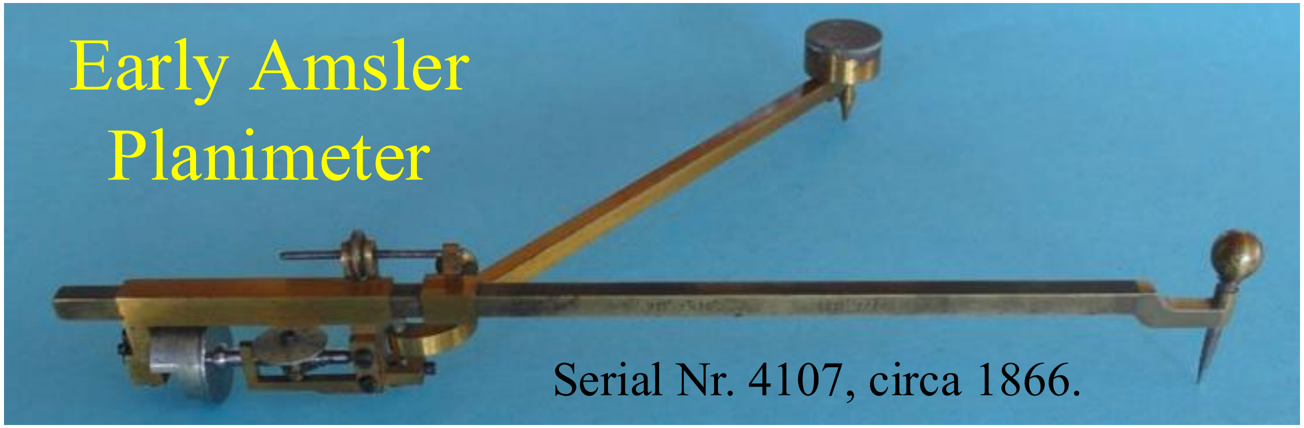
Amsler "Polar" Planimeter of 1854



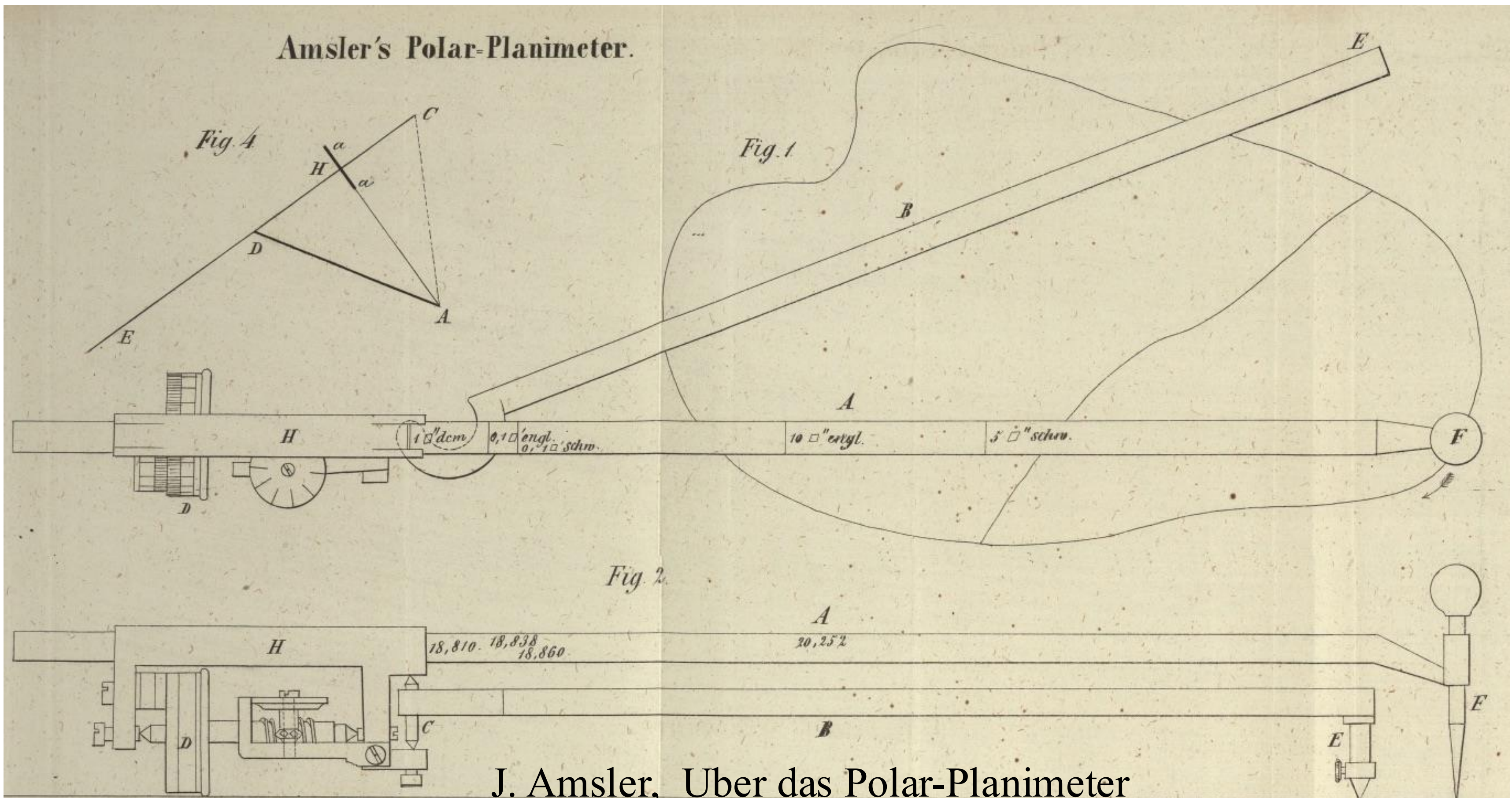
Two new features:

- Indicator Dial that rolls in one direction activated by rolling over the paper and slips in the other.
- Linkage between the tracer point at the end of the measuring arm and a fixed pole, held in place by a needle point, forcing the hinge to travel along an arc of a circle.

Early Amsler Planimeter

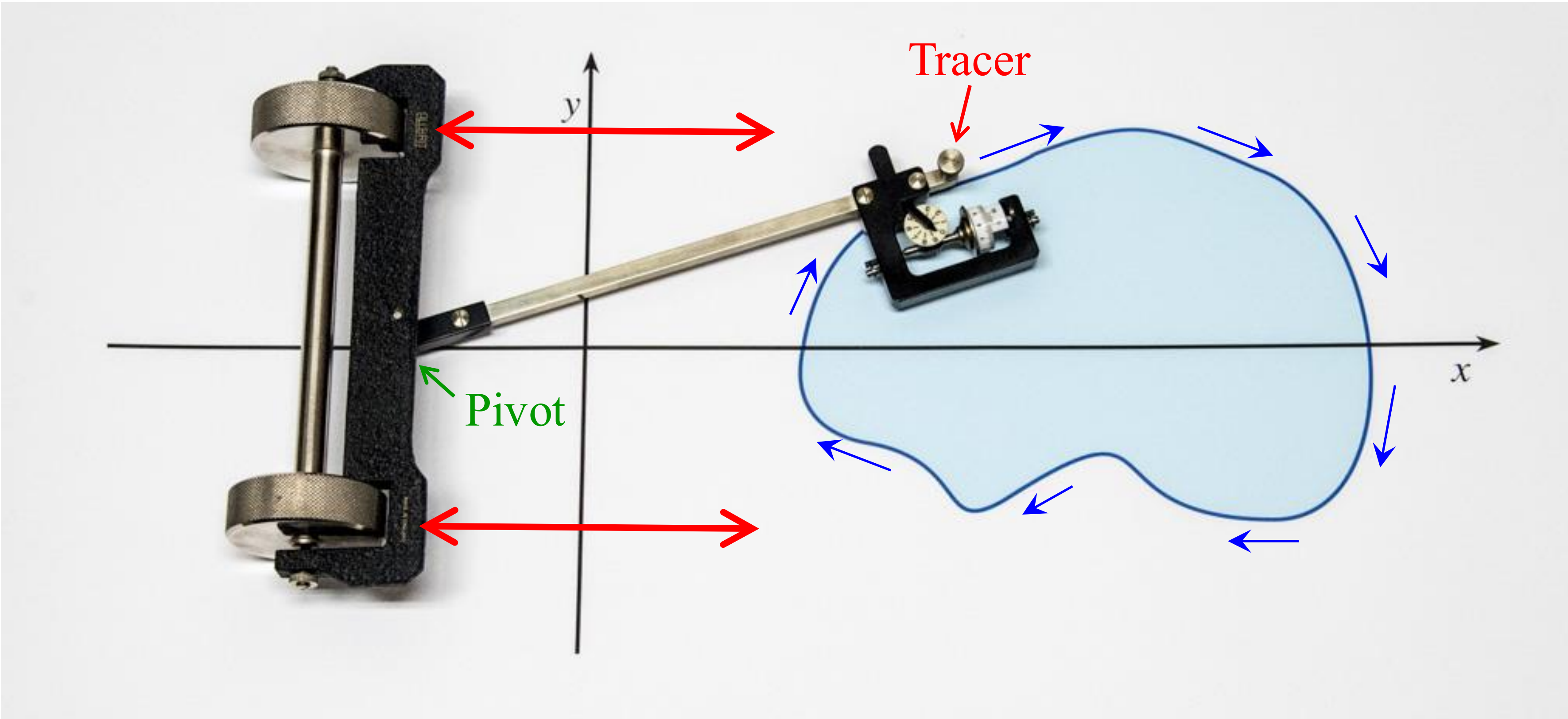


Serial Nr. 4107, circa 1866.



J. Amsler, Uber das Polar-Planimeter

Dinglers polytechnisches Journal, 1856, Band 140, Nr. LXXIII, (S. 321-327)



Allbrit Linear Rolling Planimeter

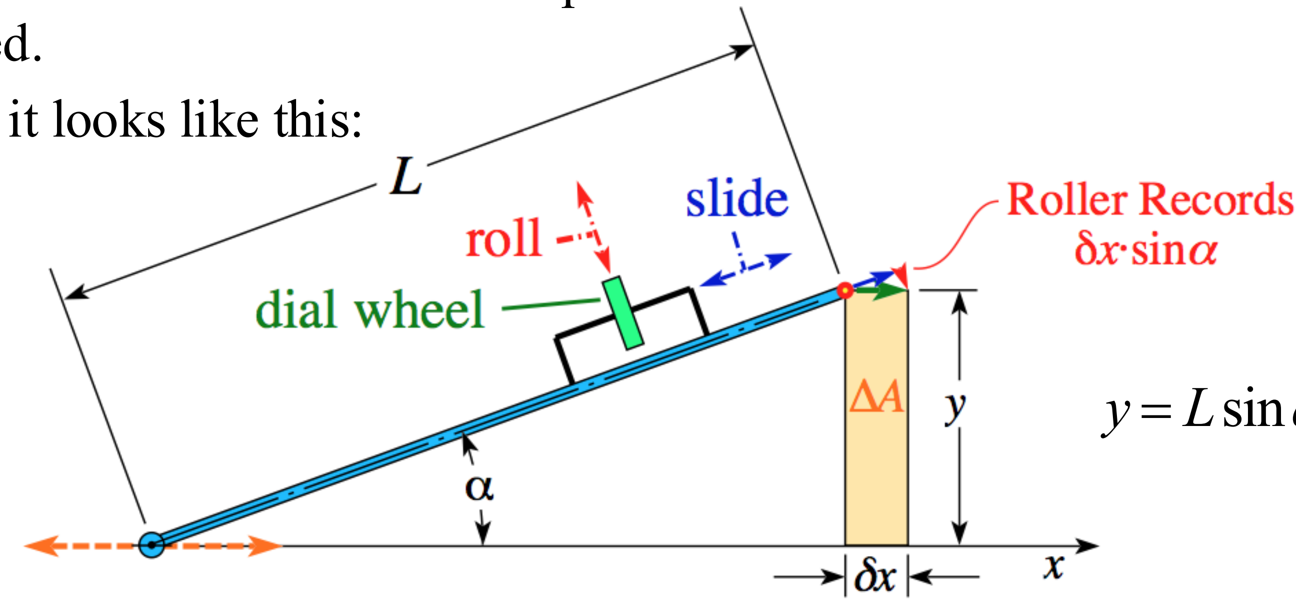
This device dates from the early 20th century. The rolling/sliding dial mechanism this planimeter uses is derived from Amsler's mechanism, invented half a century earlier. It is shown here because its operation is simple to understand.

(photograph courtesy of Douglas Fraser, 2017)

Dial Wheel with Rolling & Sliding as Recording Mechanism

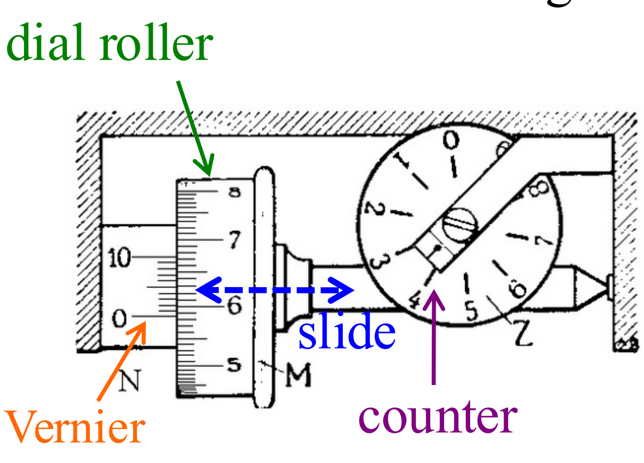
The first novel feature of Amsler's design was the use of a readout by a measuring wheel that rolls and slides as it moves while a pointer traces the outline of the shape to be measured.

In the linear geometry it looks like this:



$$y = L\sin\alpha$$

Recording $\delta x\,\sin\alpha$ is the same as recording $(y/L)\,\delta x = \Delta A/L$



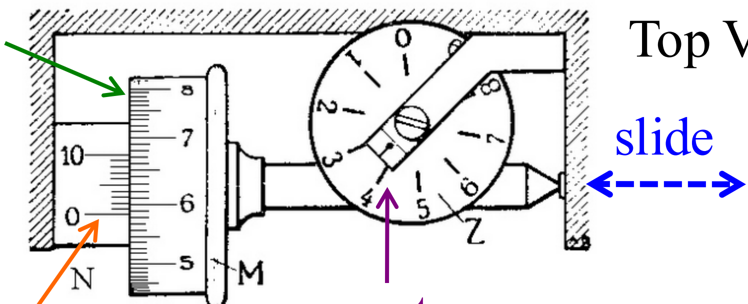
Reading shown = 3.584

$$\oint \delta x \cdot \sin\alpha = \oint \frac{y}{L}\delta x = \frac{1}{L}\oint \delta A = \frac{A}{L}$$

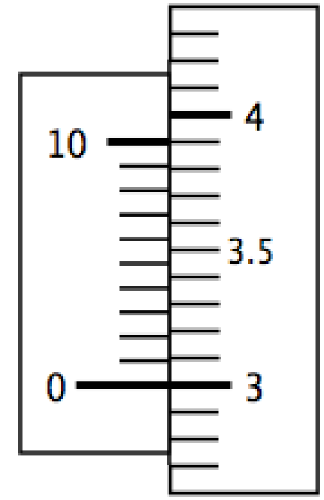
Vernier Scale

Simple method to reach an extra digit of precision.

Invented by French Mathematician Pierre Vernier (1580-1637)

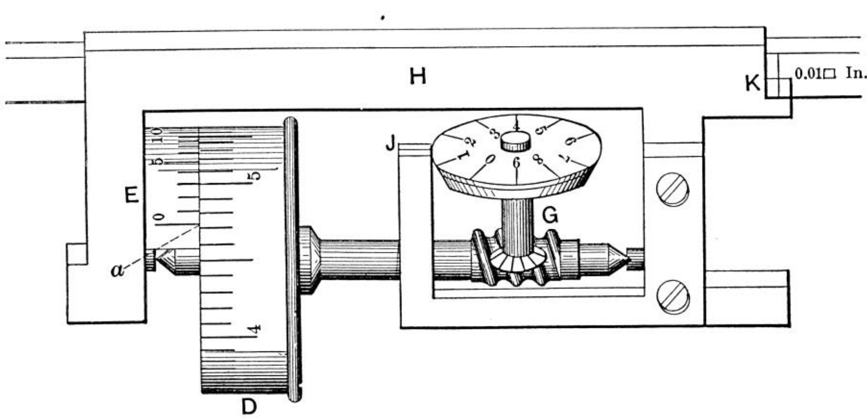


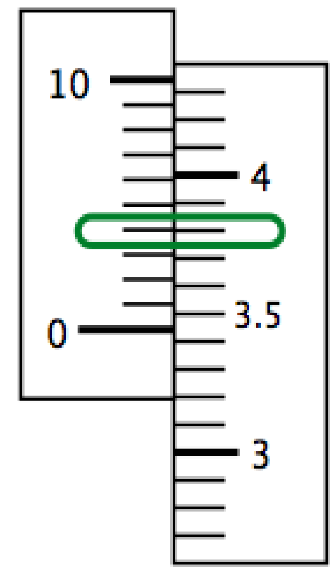
Reading shown = 3.584



10 increments left hand scale match 9 increments right hand scale.

Hidden below the counter disc is a spiral thread on the axle for the dial roller that advances the counter by one unit for each full rotation of the dial roller.

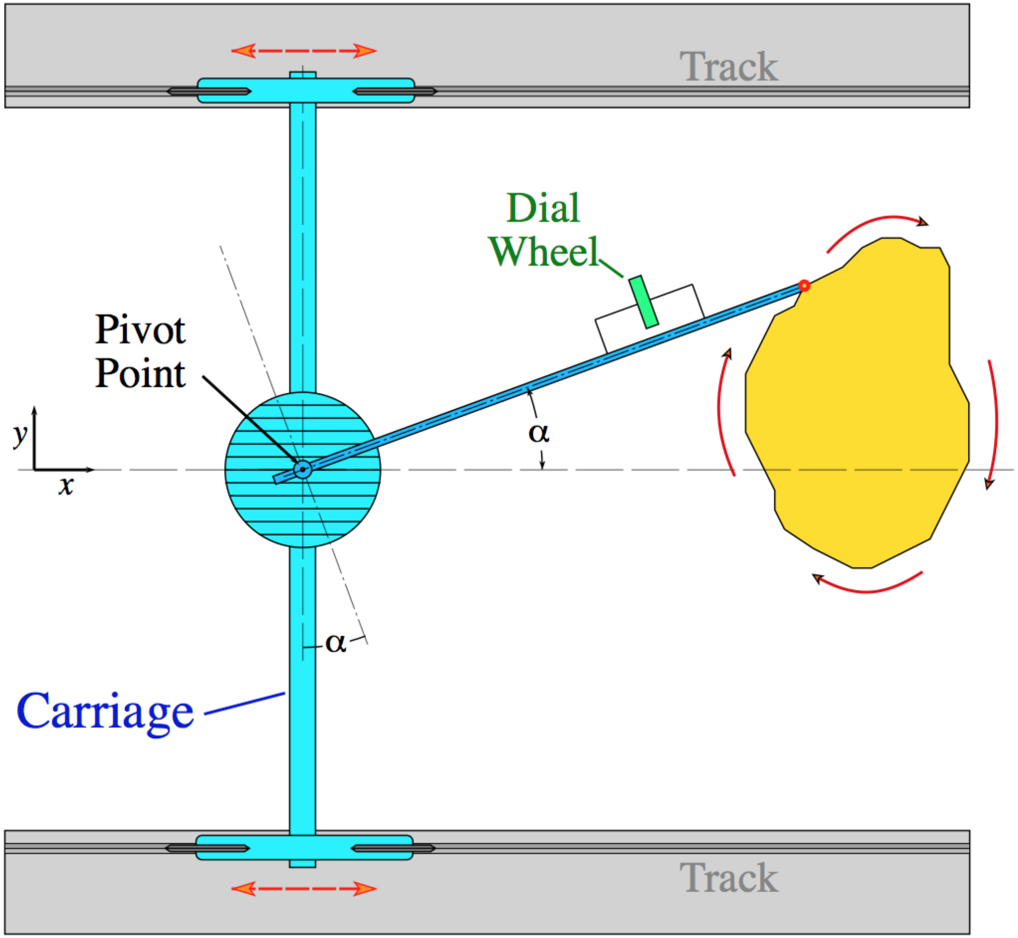


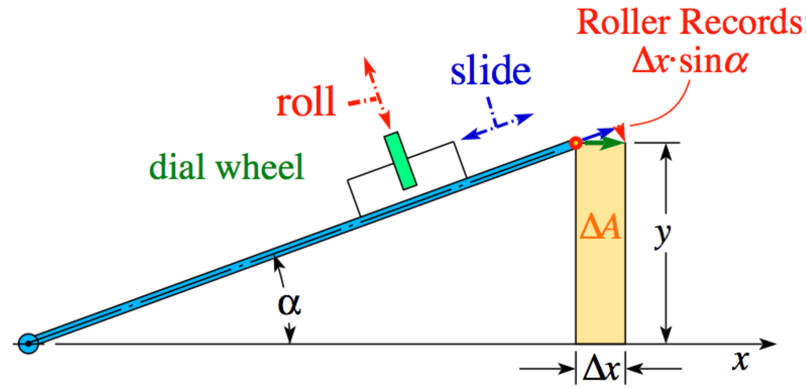


showing 3.44

(Carpenter &Diederichs, Experimental Engineering and Manual for Testing, Fig. 5)

Simple Planimeter with rolling & sliding dial wheel.





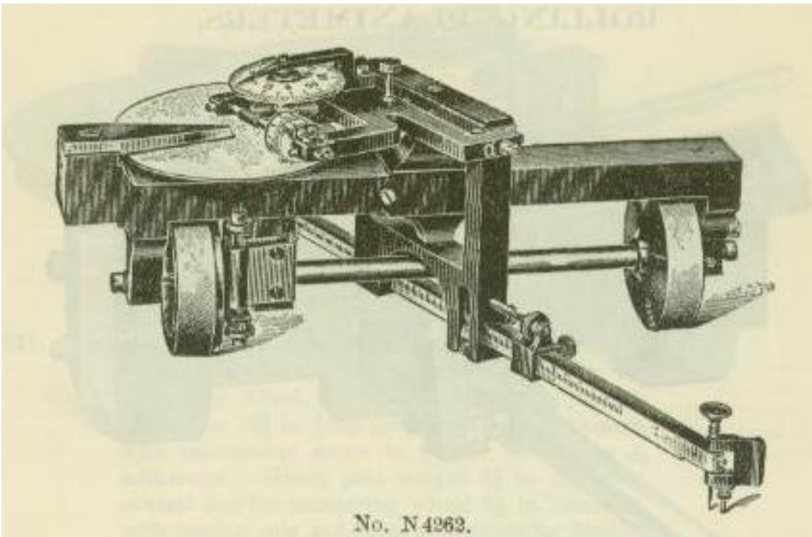
$$y = L\sin\alpha$$

Recording $\Delta x\ \sin\alpha$
is the same as
recording $(y/L)\ \Delta x$
$= \Delta A/L$

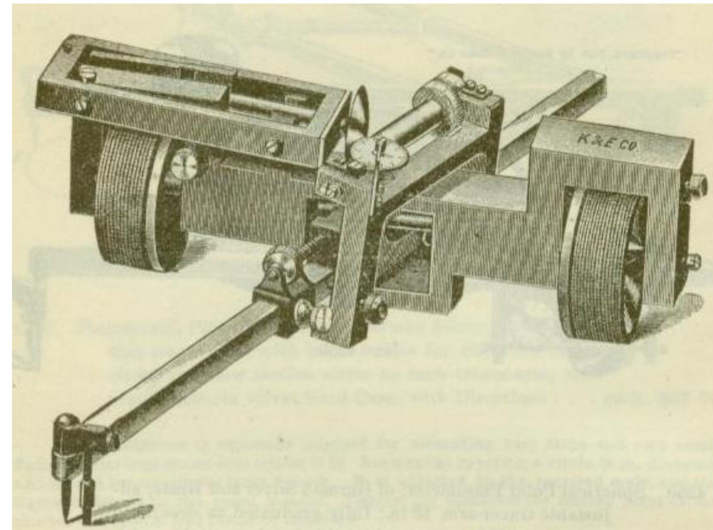
$$\oint \delta x \cdot \sin\alpha = \oint \delta x \cdot \sin\alpha = \oint \frac{y}{L}\delta x = \frac{1}{L}\oint \delta A = \frac{A}{L}$$

Other Linear Planimeter mechanisms to measure increments of $y \cdot \delta x$

("Linear" in the sense that on the rollers the mechanism moves along a straight line.)



Recording on a rotating disk.



Recording on a rotating spherical cap.

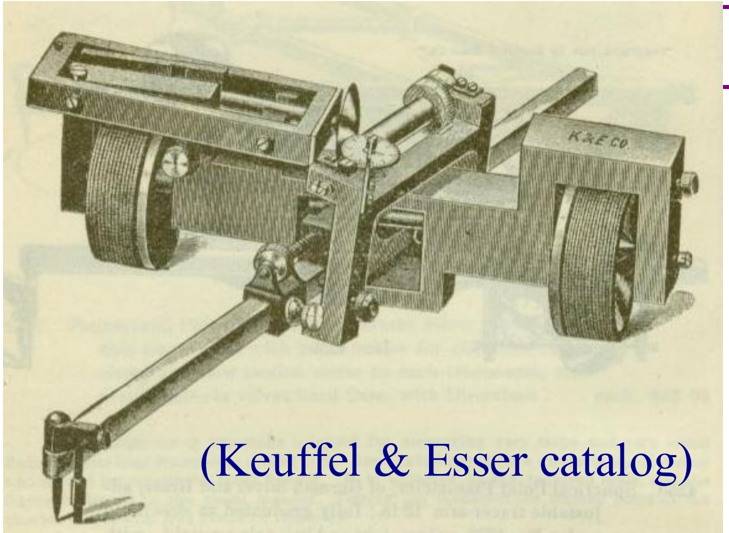
Coradi Rolling Planimeters

(sketches from Keuffel & Esser catalog)

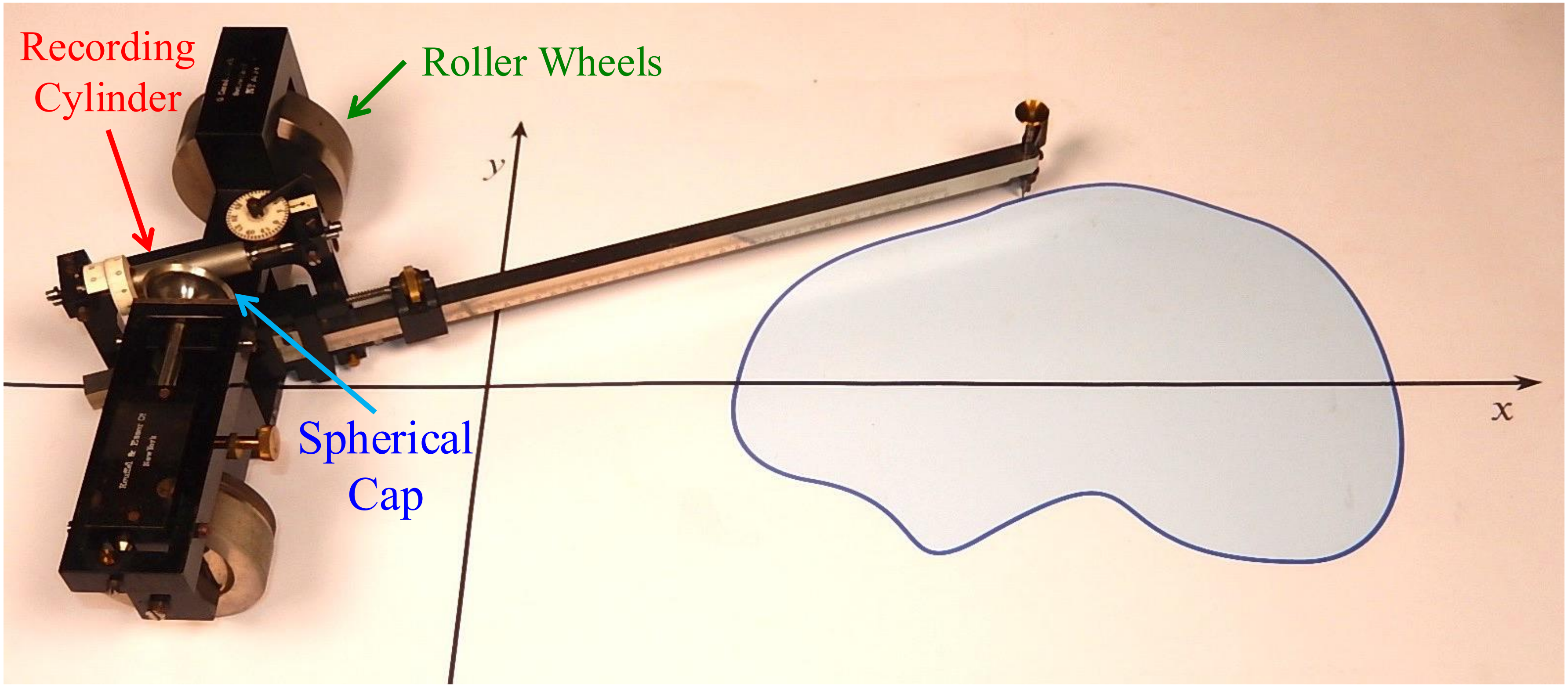
Coradi Rolling Sphere Planimeter

Linear planimeter mechanism to measure increments of $y \cdot \delta x$

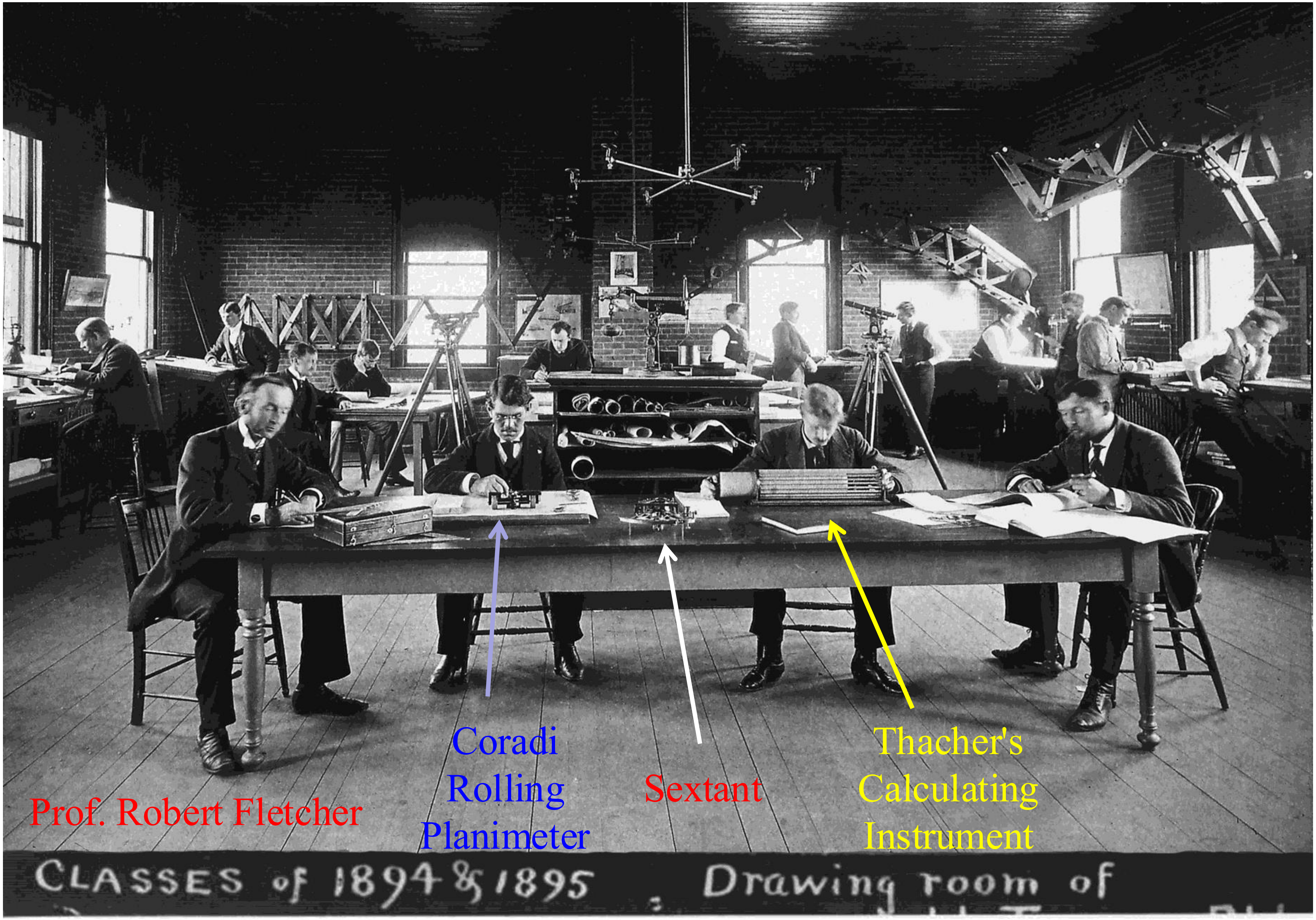
Recording through a spherical cap geared to rotate with the rotation of the roller wheels.



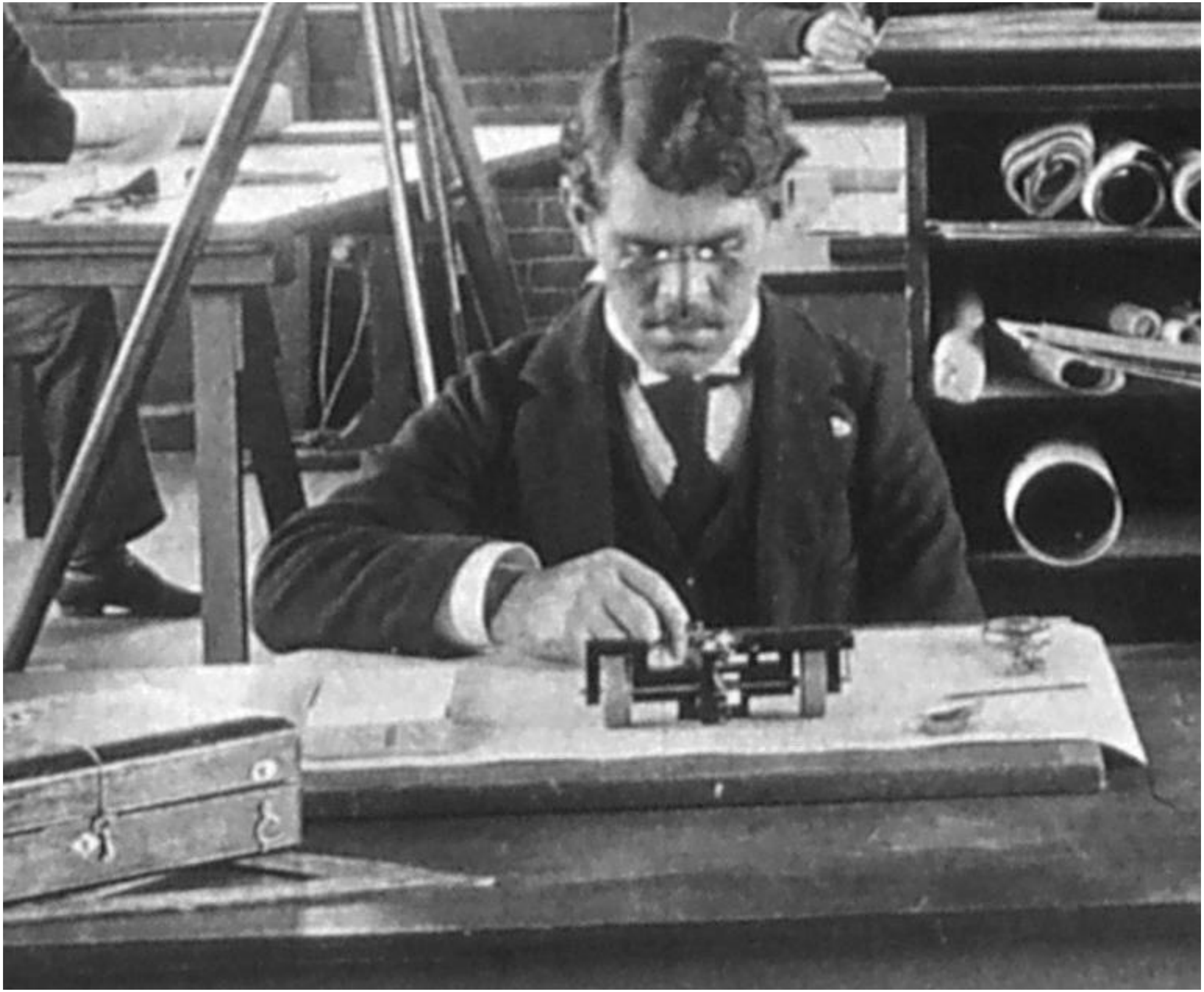
(Keuffel & Esser catalog)



Computing Gadgets for the Late Victorian Engineering School



(Photograph from Dartmouth College Archives)

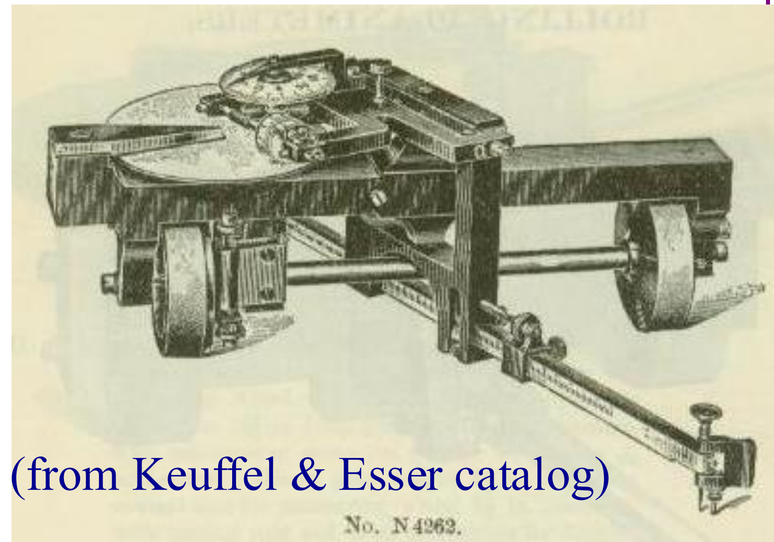


Thayer School's Rolling Sphere Planimeter

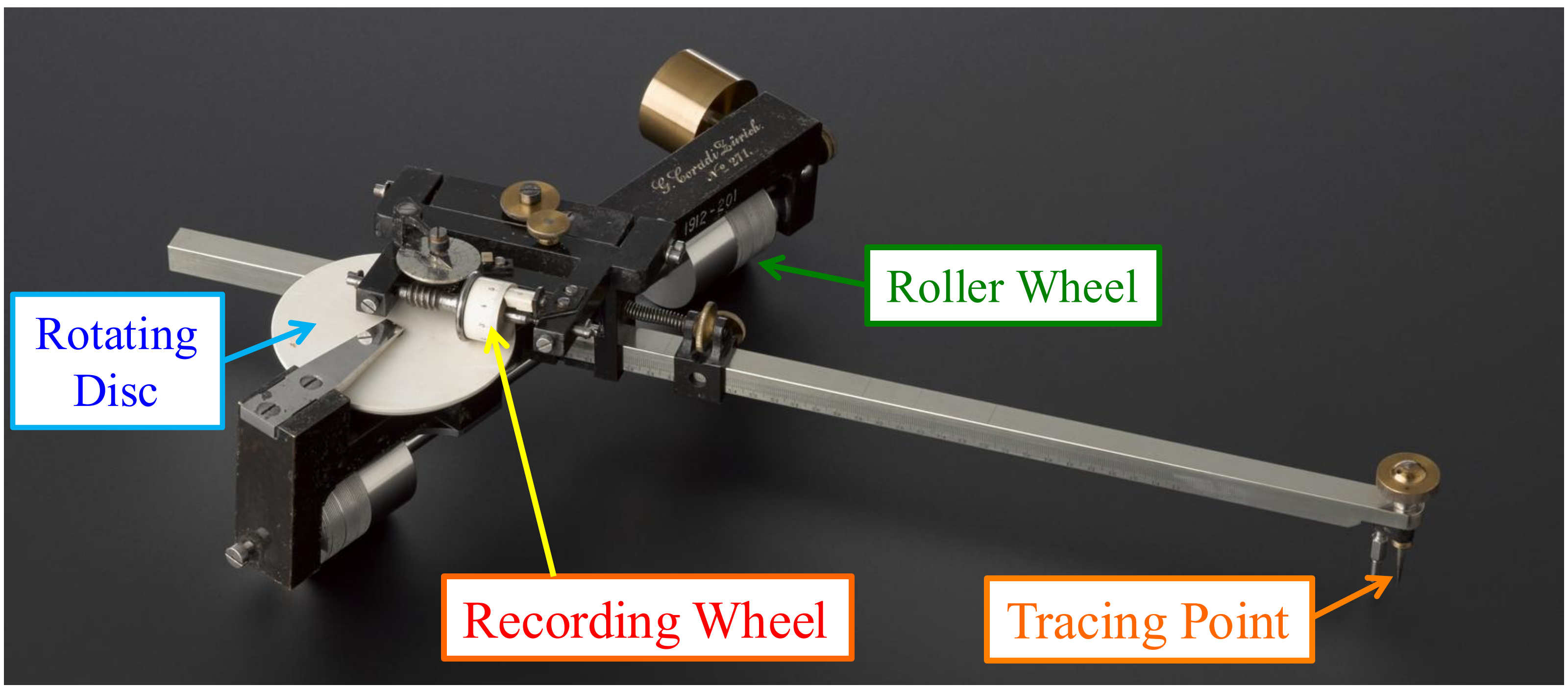
Coradi Rolling Disc Planimeter

Linear planimeter mechanism to measure increments of $y \cdot \delta x$

Recording on a rotating disc that is geared to rotate with the roller wheels. The recording wheel slips and rolls on the rotating disc according to its location that changes according to the angle of the tracing arm.

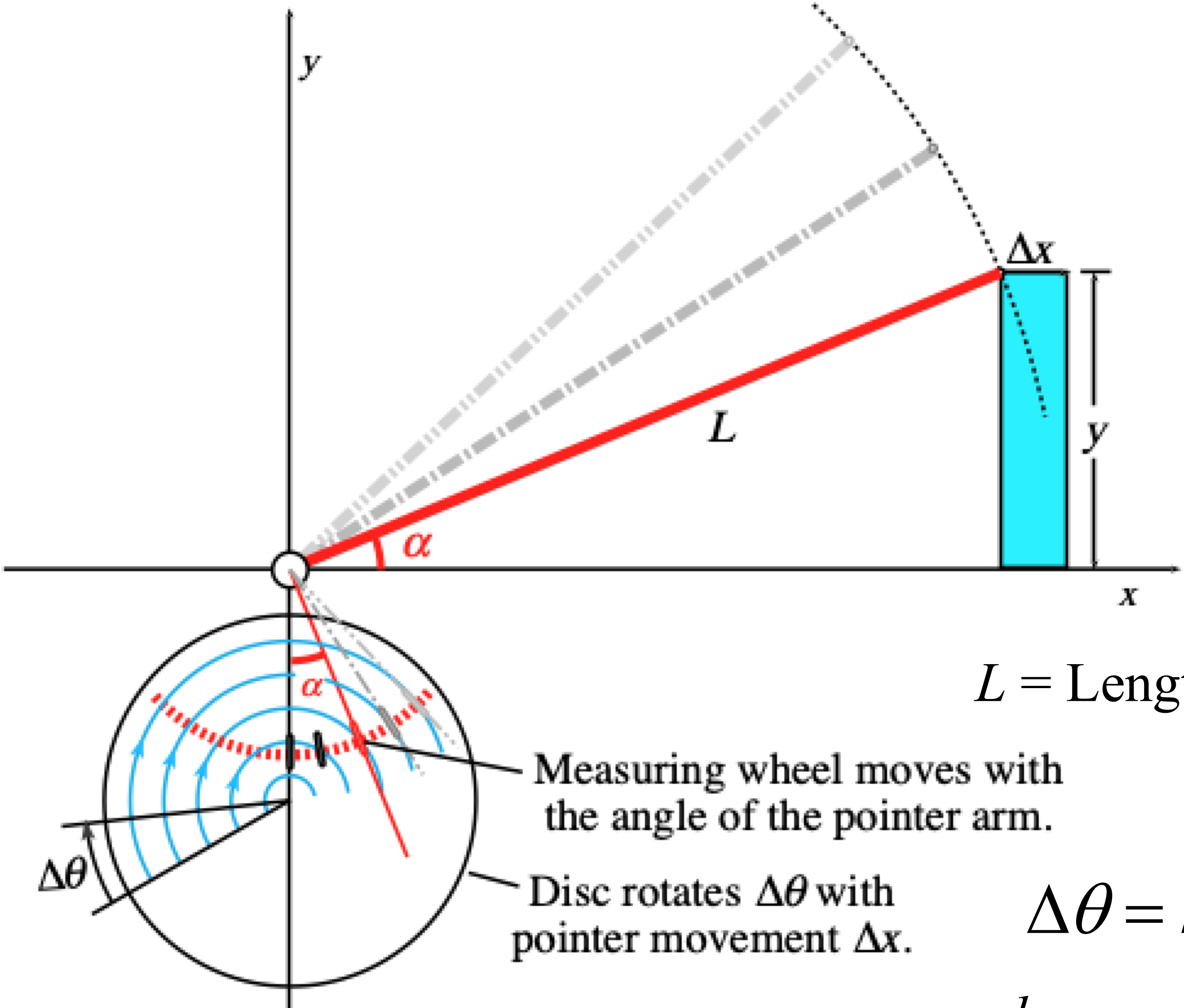


(from Keuffel & Esser catalog)



(Planimeter was sold in the U.S. as K&E model 4262;
photo from smgco-images.s3.amazonaws.com/media/W/P/A/large_1912_0201_0001_.jpg)

Rotating Disc Measurement Mechanism



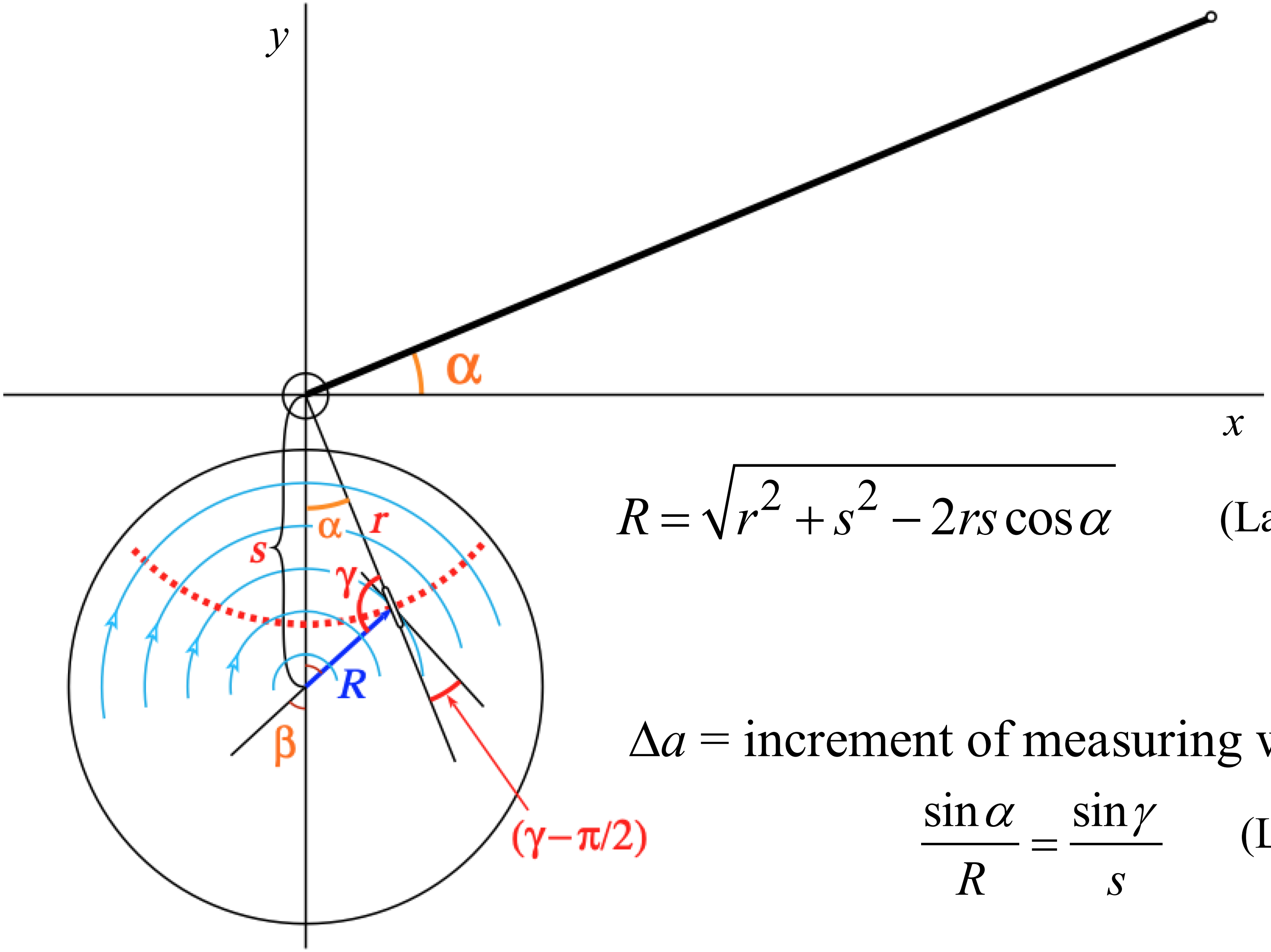
$$\Delta A = L\Delta x \sin\alpha$$

L = Length of Pointer Arm

$$\Delta\theta = k\Delta x$$

k = gearing constant for disc rotation

Rotating Disc Measurement Geometry



$$R = \sqrt{r^2 + s^2 - 2rs\cos\alpha}$$ (Law of Cosines)

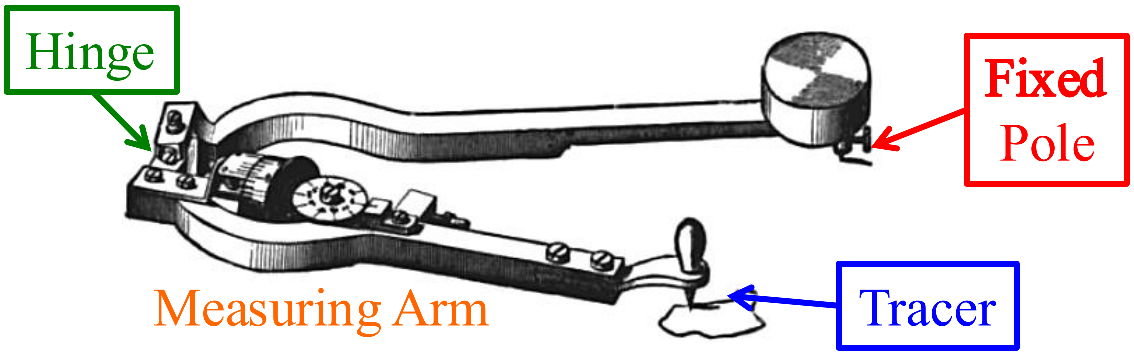
Δa = increment of measuring wheel travel.

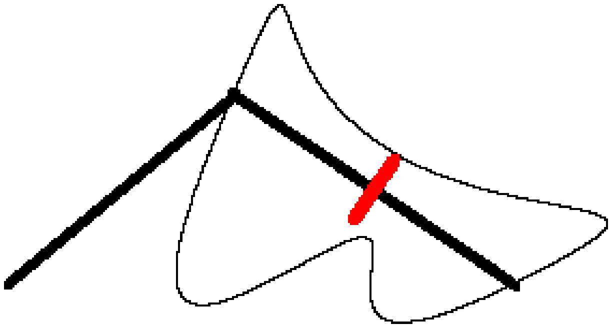
$$\frac{\sin\alpha}{R} = \frac{\sin\gamma}{s}$$ (Law of Sines)

Polar Planimeters

The second novel feature of Amsler's invention the simple linkage of two arms hinged together:

The rolling/sliding measurement system works for any shape of the track the pivoting end of the measuring arm must follow.





Jakob Amsler-Laffon

(1823 – 1912)
Swiss Mathematician and Industrialist
Inventor of the Polar Planimeter

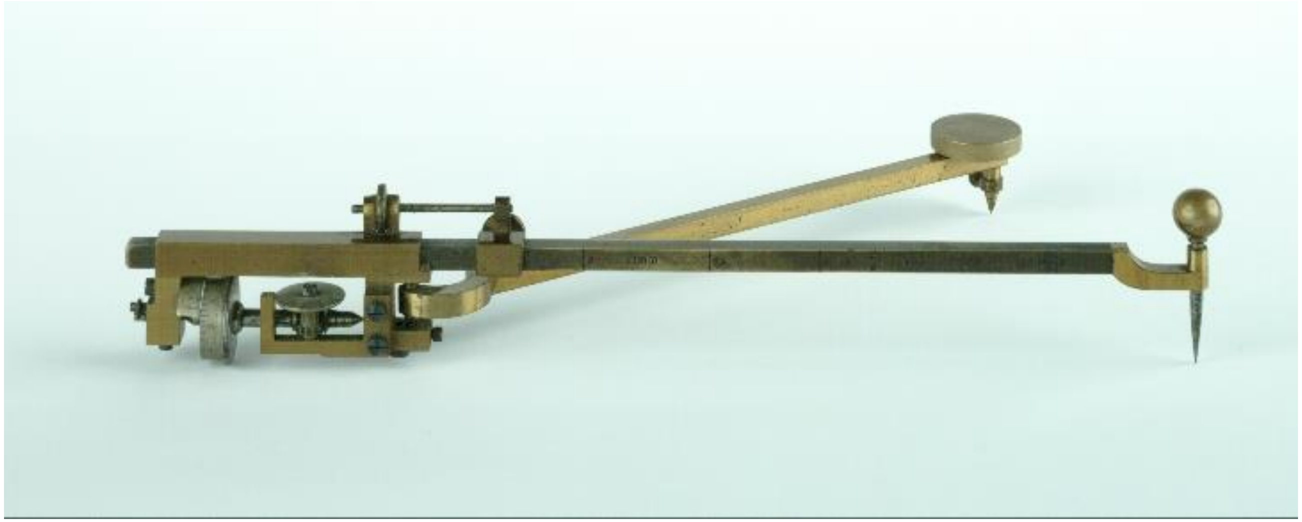
J. Amsler *Über die mechanische Bestimmung des Flächeninhaltes,..insbesondere über einen neuen Planimeter* 1856

Animation: http://persweb.wabash.edu/facstaff/footer/Planimeter/Polar&Linear.htm; by Prof. Robert Foote

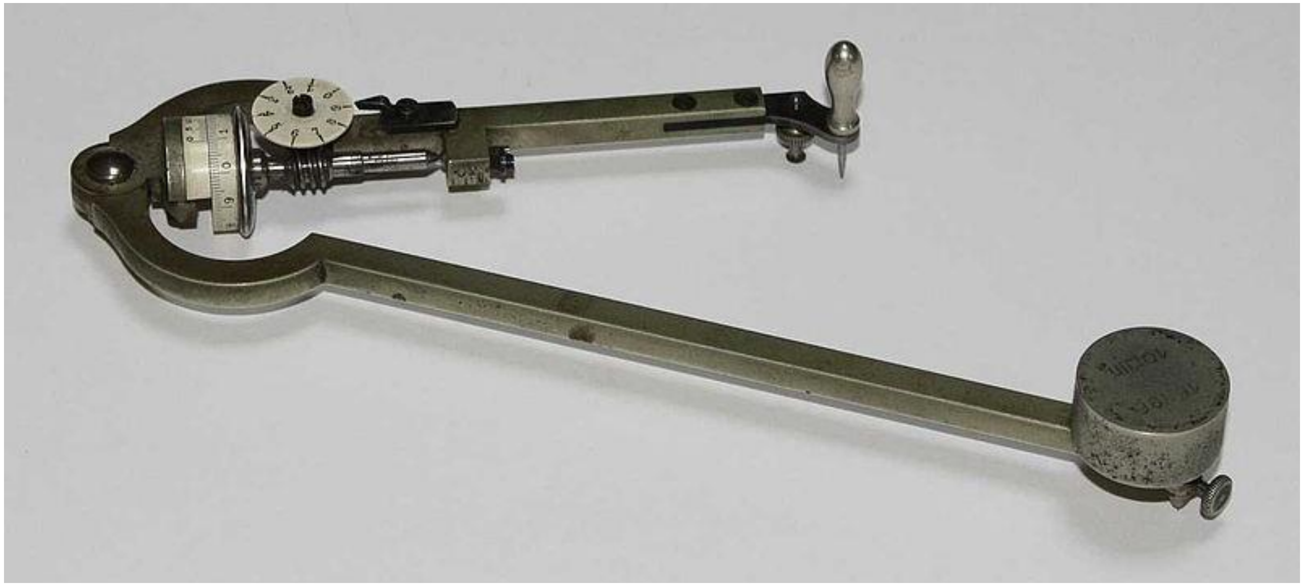
Figure from Keuffel & Esser catalog, 1908, shown in Wikipedia, *Planimeter,* (2017).

(Portrait: Wikipedia, *Jakob Amsler-Laffon*, 2017)

Amsler Polar Planimeters



(http://www.history.didaktik.mathematik.uni-wuerzburg.de/ausstell/planimet/amsler_7139.html)

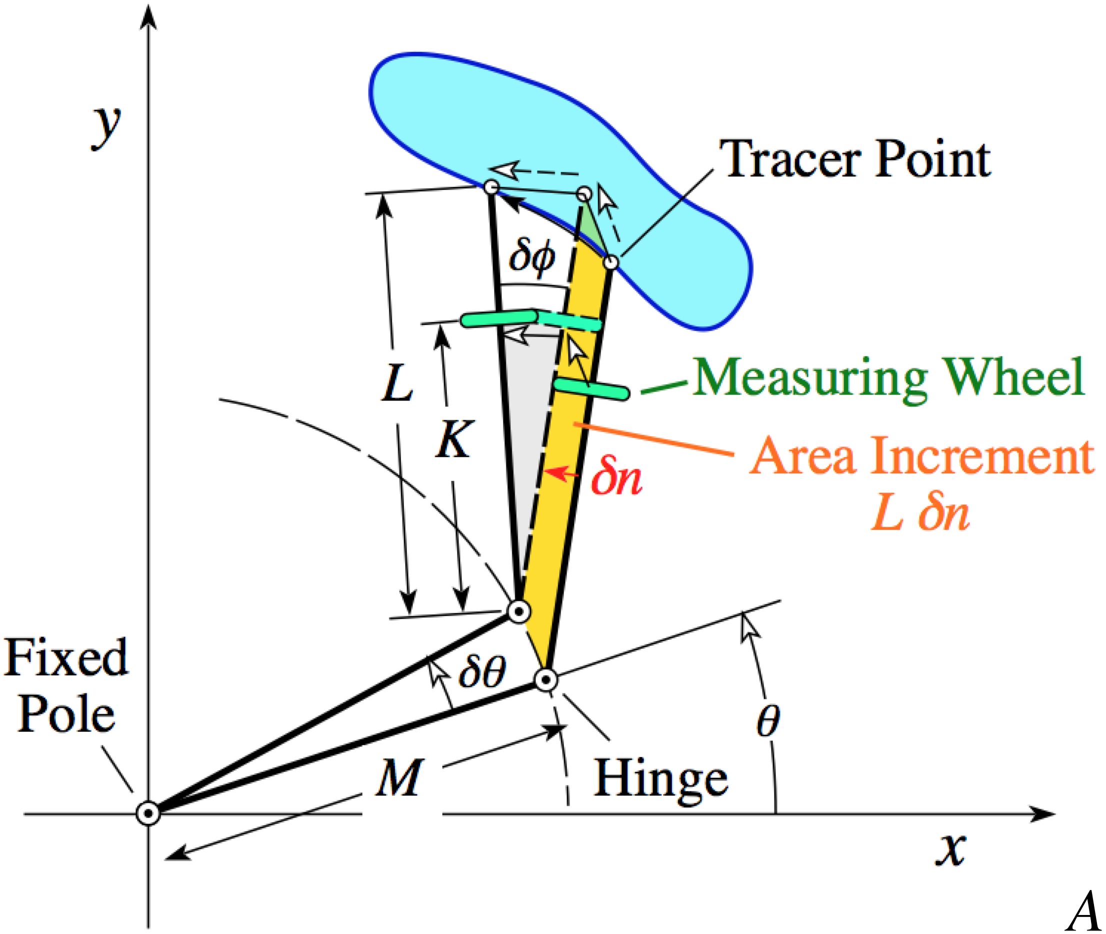


(Wikipedia, *Planimeter*, Creative Commons, creator Nol Aders)

The planimeters became a standard office instrument.

From the National Archives. https://www.flickr.com/photos/usnationalarchives/6789717264/

Operation of the Polar Planimeter



(The $\delta\phi$ increments give no net contribution.)

Decompose the area traversed by the measuring arm into

- area increments due to $\delta\phi$.
- area increments due to $\delta\theta$.

$$dA = \frac{1}{2} L^2 \delta\phi + L\delta n$$

Distance measured by the roller:

$$ds = \delta n + K\delta\phi$$

$$\delta n = ds - K\delta\phi$$

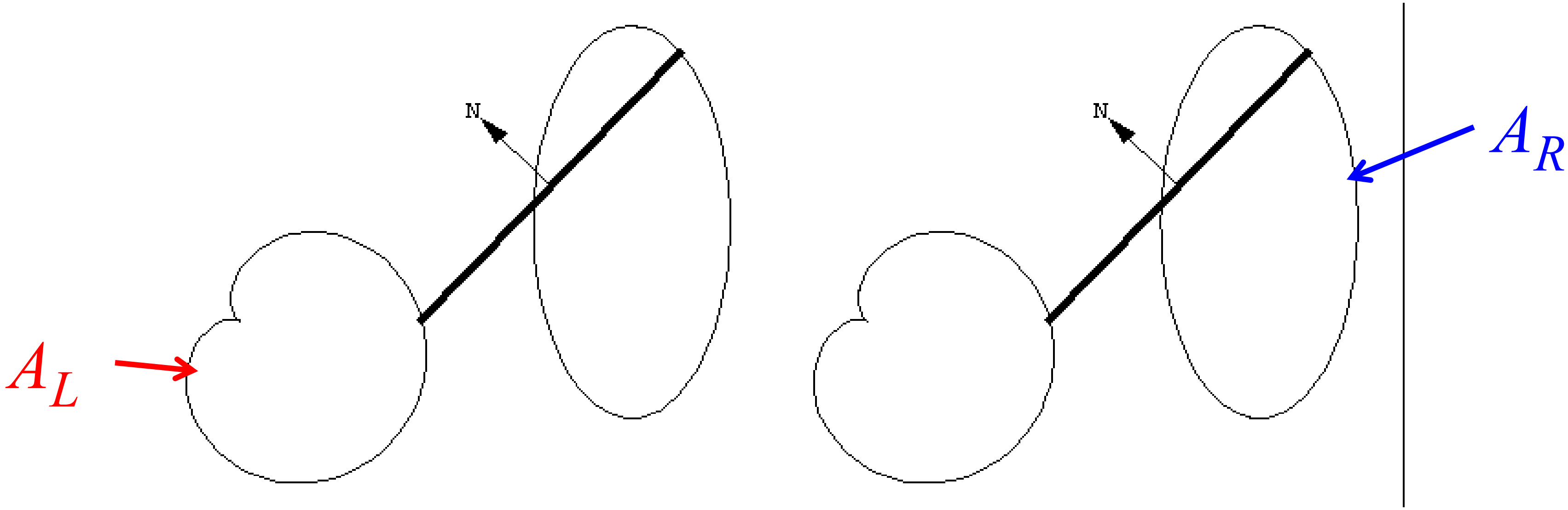
$$dA = \frac{1}{2} L^2 \delta\phi + L ds - LK\delta\phi$$

$$A = \oint L\, ds + \underbrace{\frac{1}{2}\oint L^2 \delta\phi - \oint LK\delta\phi}_{\left(\oint \delta\phi = 0\right)}$$

$$\boxed{A = L\oint ds = LS}$$

Area Difference Theorem

A moving line segment sweeps out area, called positive as the line moves in the direction of its normal vector N, and negative if it moves in the opposite direction.



Positive area is shown in blue, negative area in red.
The left-hand region (of area A_L) is swept in the negative direction.
The right-hand region (of area A_R) is swept in the positive direction.
The region between the two areas is swept both positive and negative directions.

The total signed area swept out by the line segment is $A_R - A_L$.

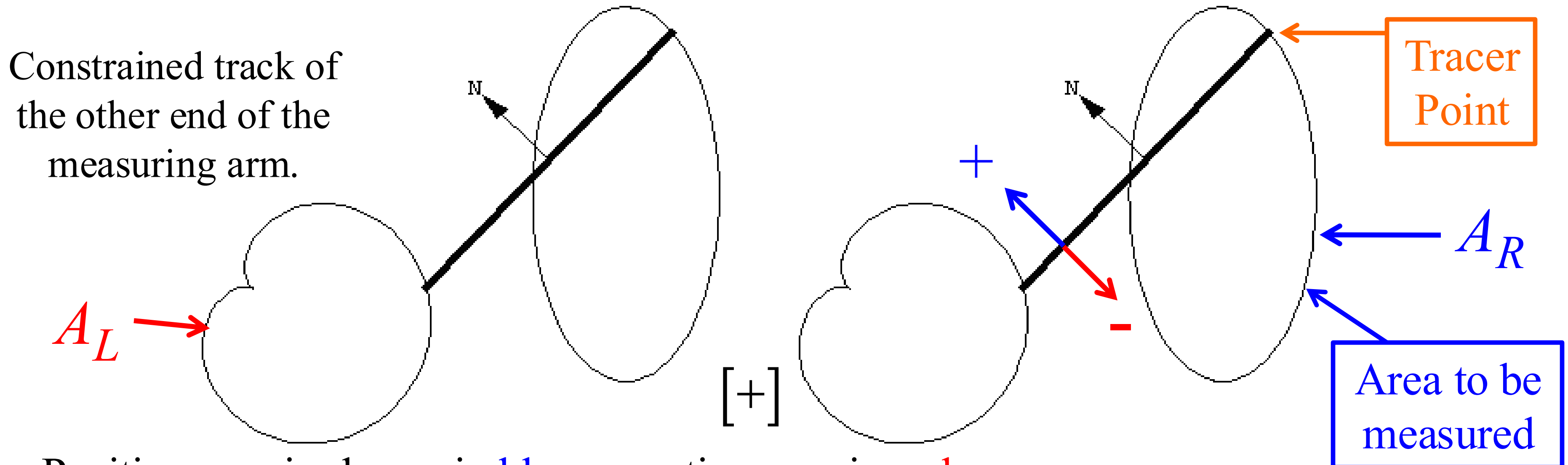
For a planimeter the left edge of the line moves back and forth along a fixed track:

$\Rightarrow A_L = 0.$

Animation by Prof. Robert Foote from
http://persweb.wabash.edu/facstaff/footer/Planimeter/AreaDiffThm.htm

Area Difference Theorem

A moving line segment sweeps out area, called positive as the line moves in the direction of its normal vector N, and negative if it moves in the opposite direction.



Positive area is shown in blue, negative area in red.
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Animation from Prof. Robert Foote
http://persweb.wabash.edu/facstaff/footer/Planimeter/AreaDiffThm.htm

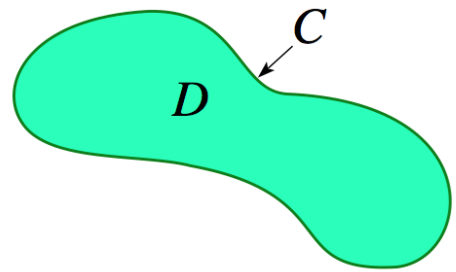
Green's Theorem for Planimeters

The validity of the planimeter may be verified by applying Green's Theorem*.

Consider a region *D* surrounded by closed curve *C* .

Let *P(x,y)* and *Q(x,y)* be functions with continuous first derivatives over *D*.

(We may think of *(P,Q)* as a vector field.)



$$\oint_C (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

If we pick *P* and *Q* such that: $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ then the integral will give the area, *A*.

P and *Q* must correspond to the accumulated measurement of the measuring wheel as we circuit the perimeter.

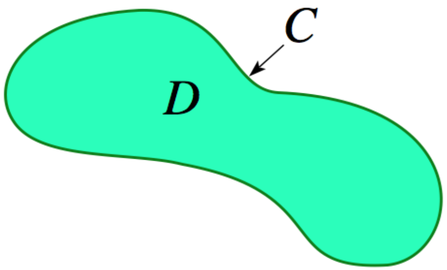
Direction of rolling is perpendicular to the measuring arm.
Measuring arm spans from point (*a*,*b*) to point on perimeter (*x*,*y*).
Vector of the measuring arm length is [*x*-*a*, *y*-*b*].
Perpendicular direction that wheel will roll: [-(*y*-*b*), (*x*-*a*)].
Moving the tracer by an increment [δx, δy]
increments the rolling by the projection of (δx, δy) in the rolling direction.

* George Green (1793 – 1841), English Baker, Miller and Mathematician.

Green's Theorem for Sliding/Rolling-Wheel Planimeters

The validity of the planimeter may be verified by applying Green's Theorem*.

(We may think of *(P,Q)* as a vector field.)



$$\oint_C (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

If we pick P and Q such that: $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ then the integral will give the area, A.

P and Q must correspond to the accumulated measurement of the measuring wheel as we circuit the perimeter.

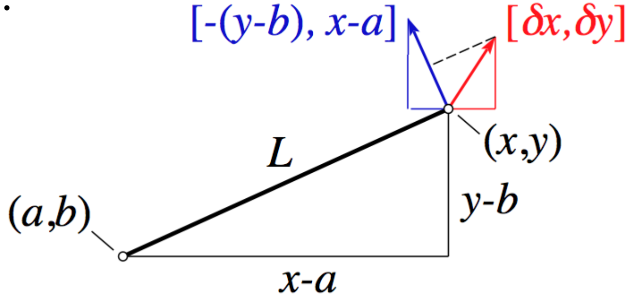
Direction of rolling is perpendicular to the measuring arm.

Measuring arm spans from point (a,b) to point on perimeter (x,y).

Vector of the measuring arm length is $[x\text{-}a,\ y\text{-}b]$.

Perpendicular direction that wheel will roll: $[-(y\text{-}b),\ (x\text{-}a)]$.

Unit Vector in the rolling direction: $\frac{1}{L}\left[-(y-b),(x-a)\right]$



Moving the tracer by an increment $[\delta x,\ \delta y]$

increments the rolling by the projection of $[\delta x,\ \delta y]$ in the rolling direction:

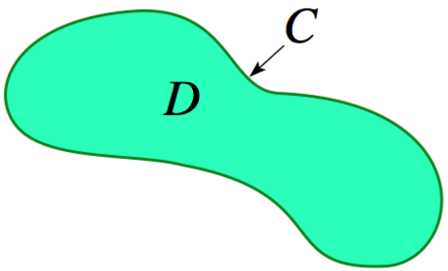
(Vector dot product) $\frac{1}{L}\left[-(y-b)\cdot\delta x+(x-a)\cdot\delta y\right]$

The scale on the rolling wheel readout is adjusted to compensate for the factor $(1/L)$,

so it reads: $\oint_C\left[-(y-b)\delta x+(x-a)\delta y\right]$

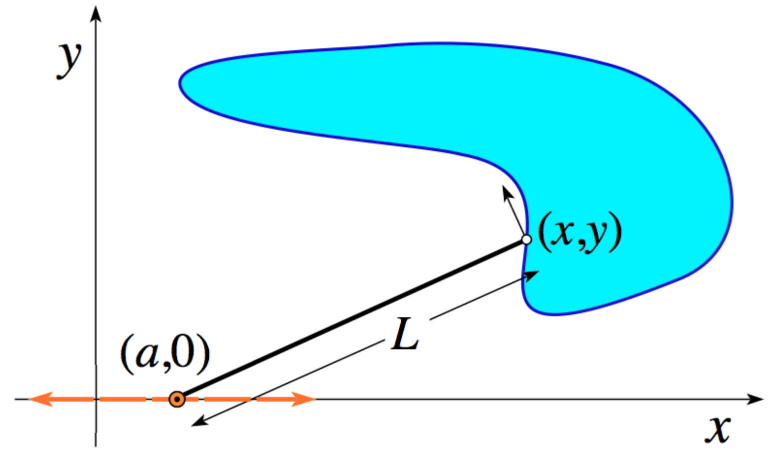
Green's Theorem for the Linear Planimeter

$$\oint_C \left(P\,dx + Q\,dy\right) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$



Pick P and Q such that:

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$



For the linear planimeter, we use: $P(x,y) = -y\ ;\quad Q(x,y) = x - a$

(Special Case of $b = 0$.)

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial (x-a)}{\partial x} - \frac{\partial (-y)}{\partial y} = \frac{\partial x}{\partial x} - \frac{\partial a}{\partial x} + \frac{\partial y}{\partial y} = 1 - \frac{\partial a}{\partial x} + 1 = 2 - \frac{\partial a}{\partial x}$$

Constraint: $(x-a)^2 + y^2 = L^2\ ;\quad (x-a) = \sqrt{\left(L^2 - y^2\right)}\ ;\quad a = x - \sqrt{\left(L^2 - y^2\right)}$

$$\Rightarrow \left(\frac{\partial a}{\partial x}\right)_y = 1 \qquad \square \qquad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2 - \frac{\partial a}{\partial x} = 2 - 1 = 1$$

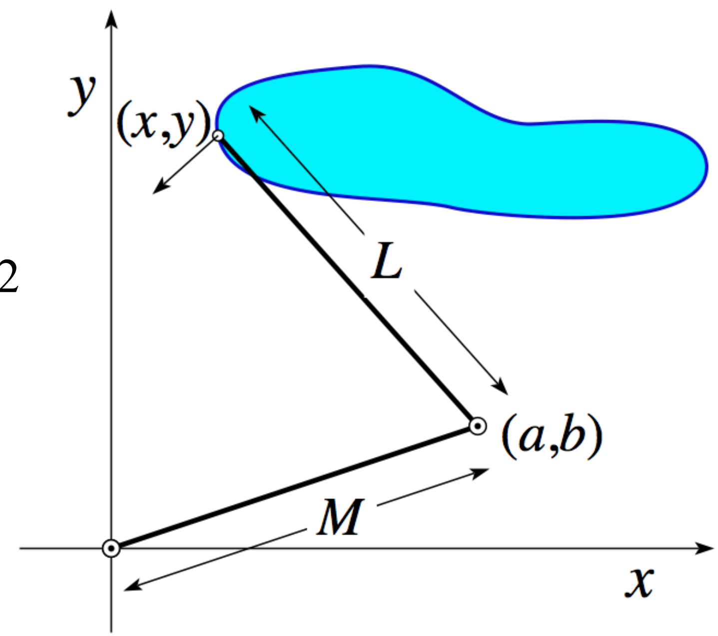
Green's Theorem for the Polar Planimeter

$$\oint_C (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

Pick *P(x,y)* and *Q(x,y)* to match the operation of the measuring wheel,

and then show that: $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ (or a constant), meaning that we measure area.

Use: $P(x,y) = -(y-b)$; $Q(x,y) = x - a$



Note that a and b, coordinates of the hinge location, are not constants, but depend on x and y.

Constraints: $a^2 + b^2 = M^2$; $(x-a)^2 + (y-b)^2 = L^2$

(Point (a,b) is constrained to a circle of radius M.)

From these:

$$\frac{\partial}{\partial x}: \quad a\frac{\partial a}{\partial x} + b\frac{\partial b}{\partial x} = 0 \; ; \quad x\frac{\partial a}{\partial x} + y\frac{\partial b}{\partial x} = x - a$$

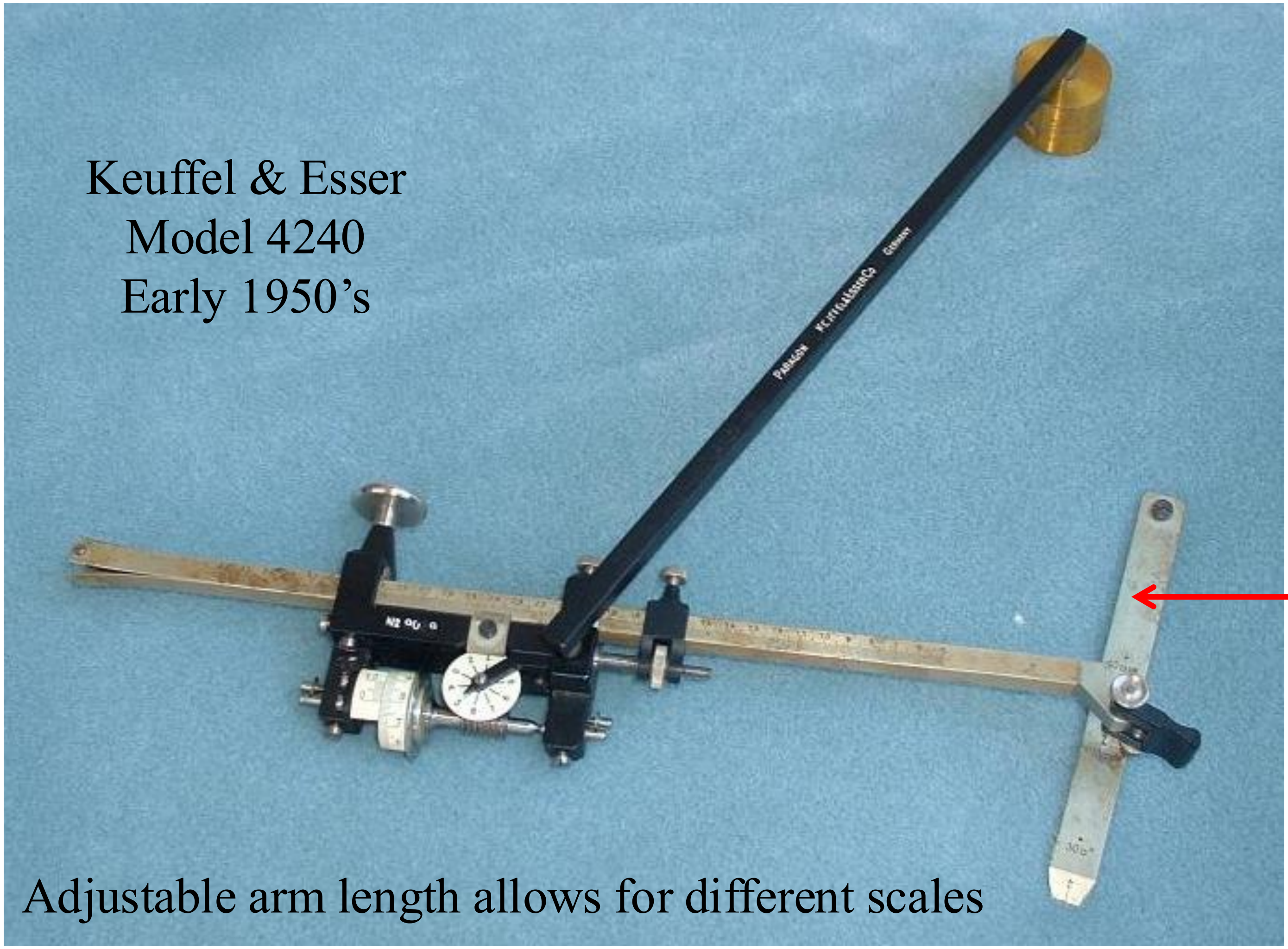
$$\frac{\partial}{\partial y}: \quad a\frac{\partial a}{\partial y} + b\frac{\partial b}{\partial y} = 0 \; ; \quad x\frac{\partial a}{\partial y} + y\frac{\partial b}{\partial y} = y - a$$

Solving 4 equations for 4 derivatives:

$$\frac{\partial a}{\partial x} = \frac{-b(x-a)}{ay-bx} \; ; \quad \frac{\partial b}{\partial x} = \frac{a(x-a)}{ay-bx} \; ; \quad \frac{\partial a}{\partial y} = \frac{-b(y-b)}{ay-bx} \; ; \quad \frac{\partial b}{\partial y} = \frac{a(y-a)}{ay-bx}$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial(x-a)}{\partial x} - \frac{\partial\left[-(y-b)\right]}{\partial y} = 2 - \frac{\partial a}{\partial x} - \frac{\partial b}{\partial y} = 2 - \left[\frac{-b(x-a)}{ay-bx} + \frac{a(y-b)}{ay-bx}\right] = 2 - \left[\frac{ay-bx}{ay-bx}\right] = 1$$

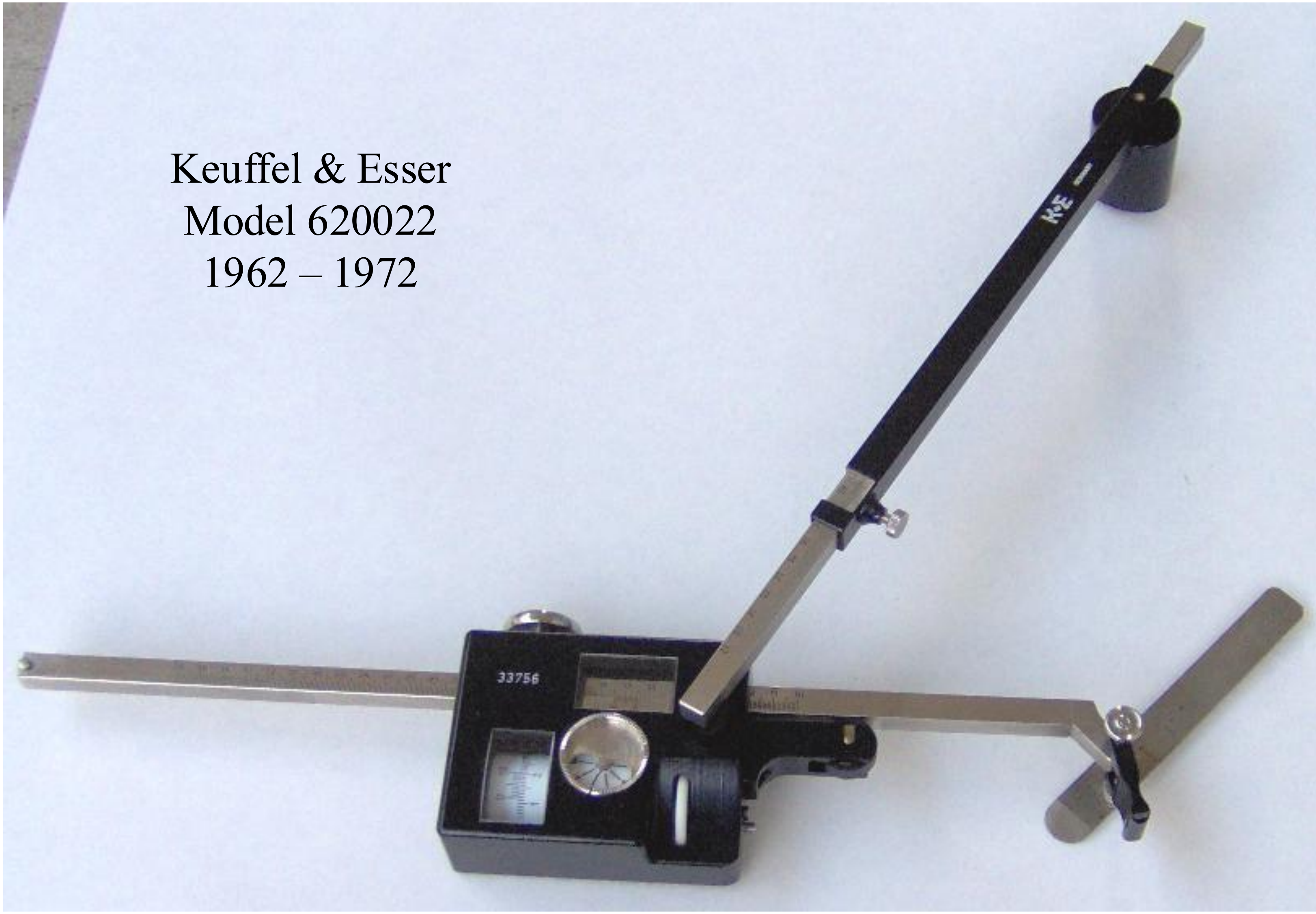
Ordinary Vintage Planimeters



Calibration Device. Beam pinned at one end allows the tracer point to be taken around a circle of known radius/area.

Collection of Clark McCoy
http://mccoys-kecatalogs.com/PlanimeterModels/PlanimeterModels.htm

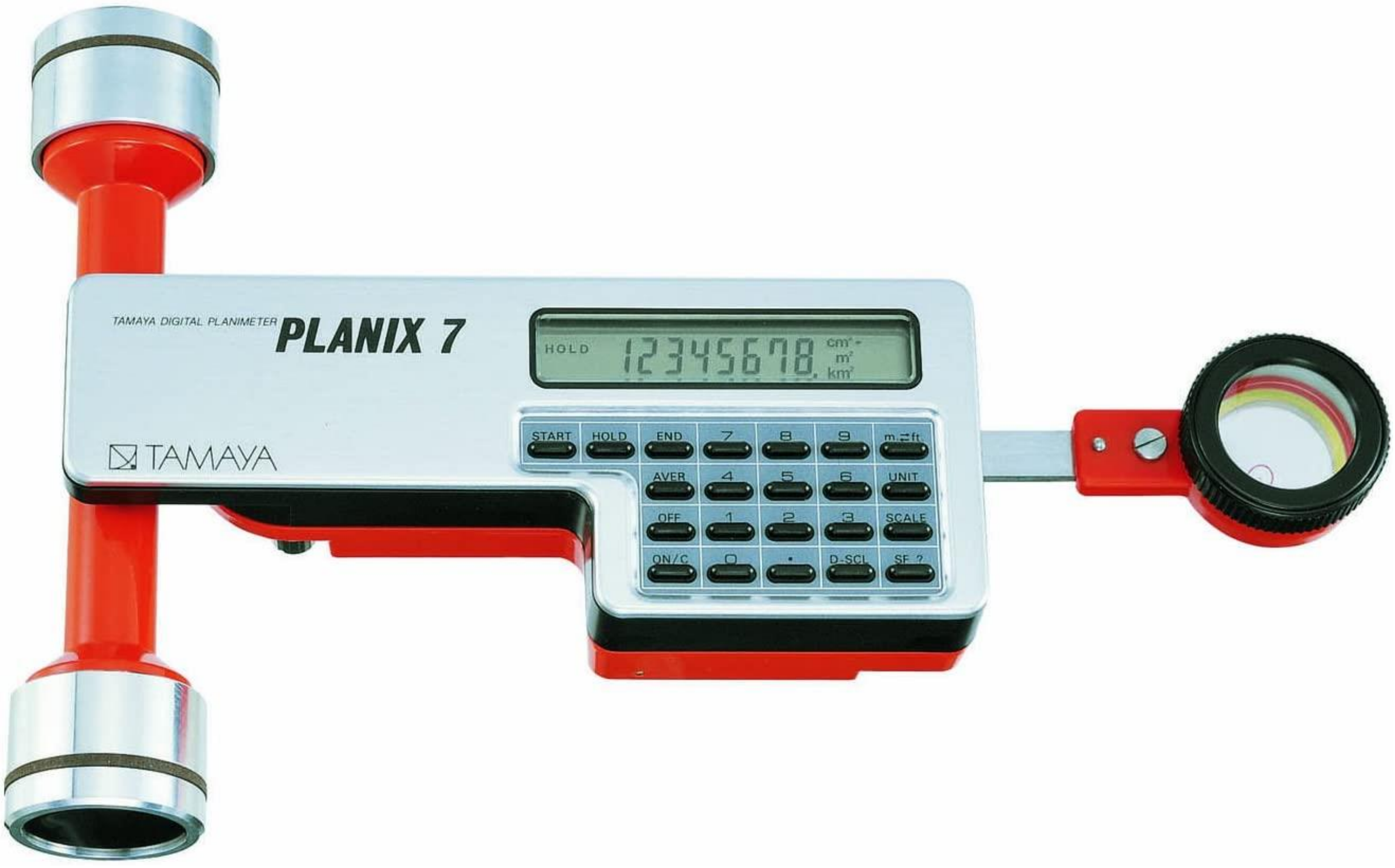
Another Vintage Planimeter



Collection of Clark McCoy
http://mccoys-kecatalogs.com/PlanimeterModels/PlanimeterModels.htm

Modern Digital Planimeters

Might possibly be purchased new.

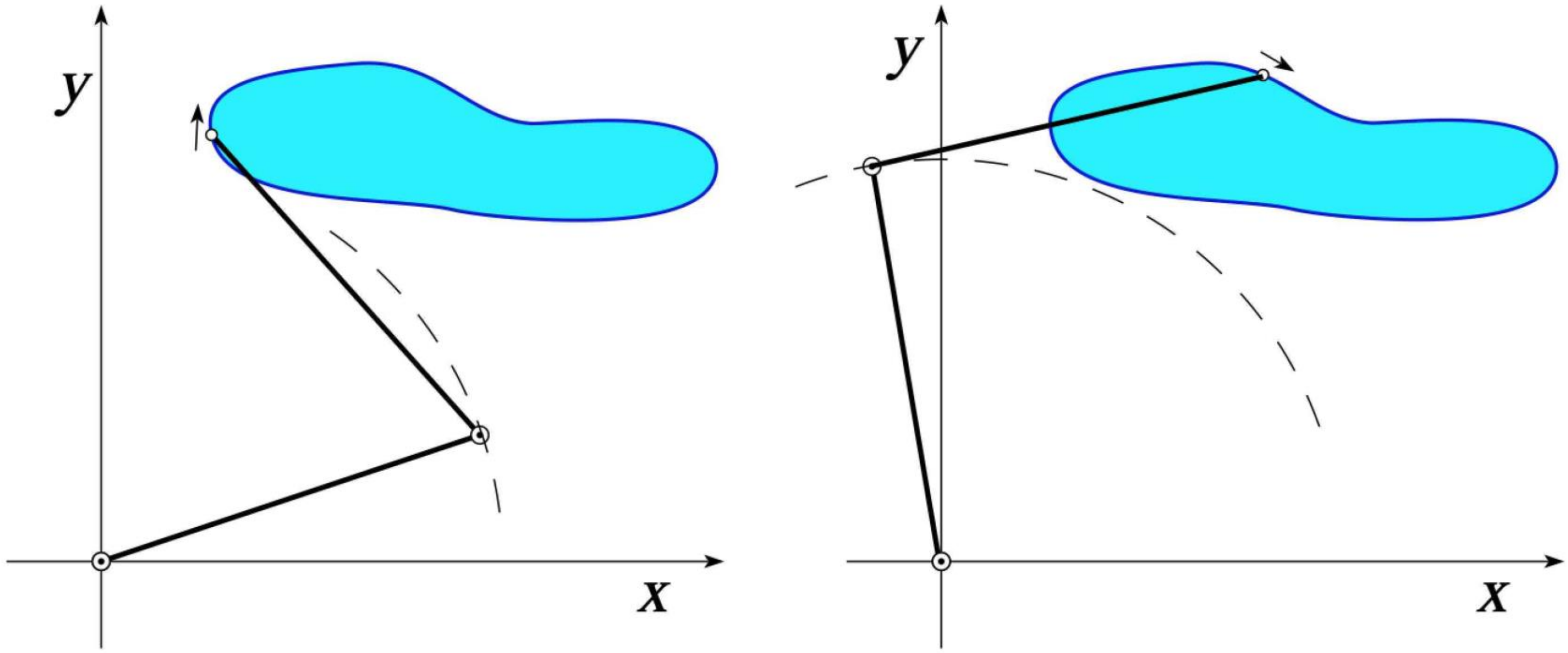


(Available at *e.g.* https://tamaya-technics.com/en/planix/)

Sources of Error

- Measuring wheel slippage.
- Incorrect arm length.
- Incorrect measuring wheel diameter.
- Misalignment of measuring-wheel axis.

  $\Rightarrow$ "Compensating" polar planimeter.

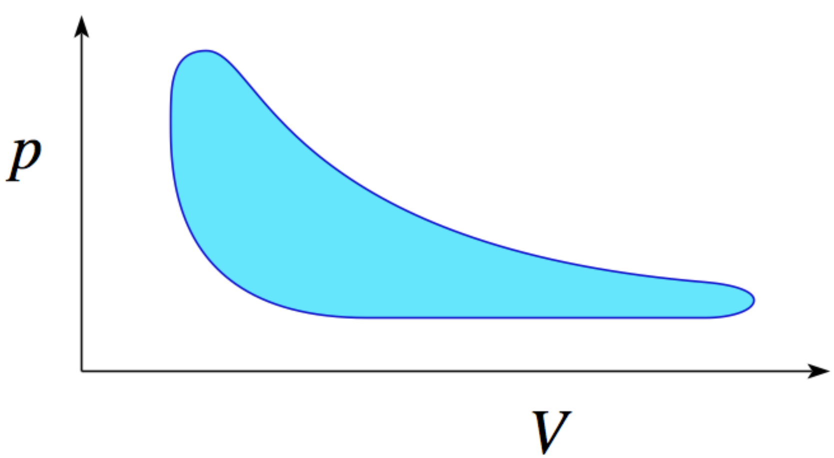


Planimeter Application for Steam Engine Testing

The power is energy/stroke × strokes/unit time.

The energy is given in terms of $\oint p\,dV$

p = Steam Pressure.
V = Volume of Steam Cylinder.
(given by distance of stroke)

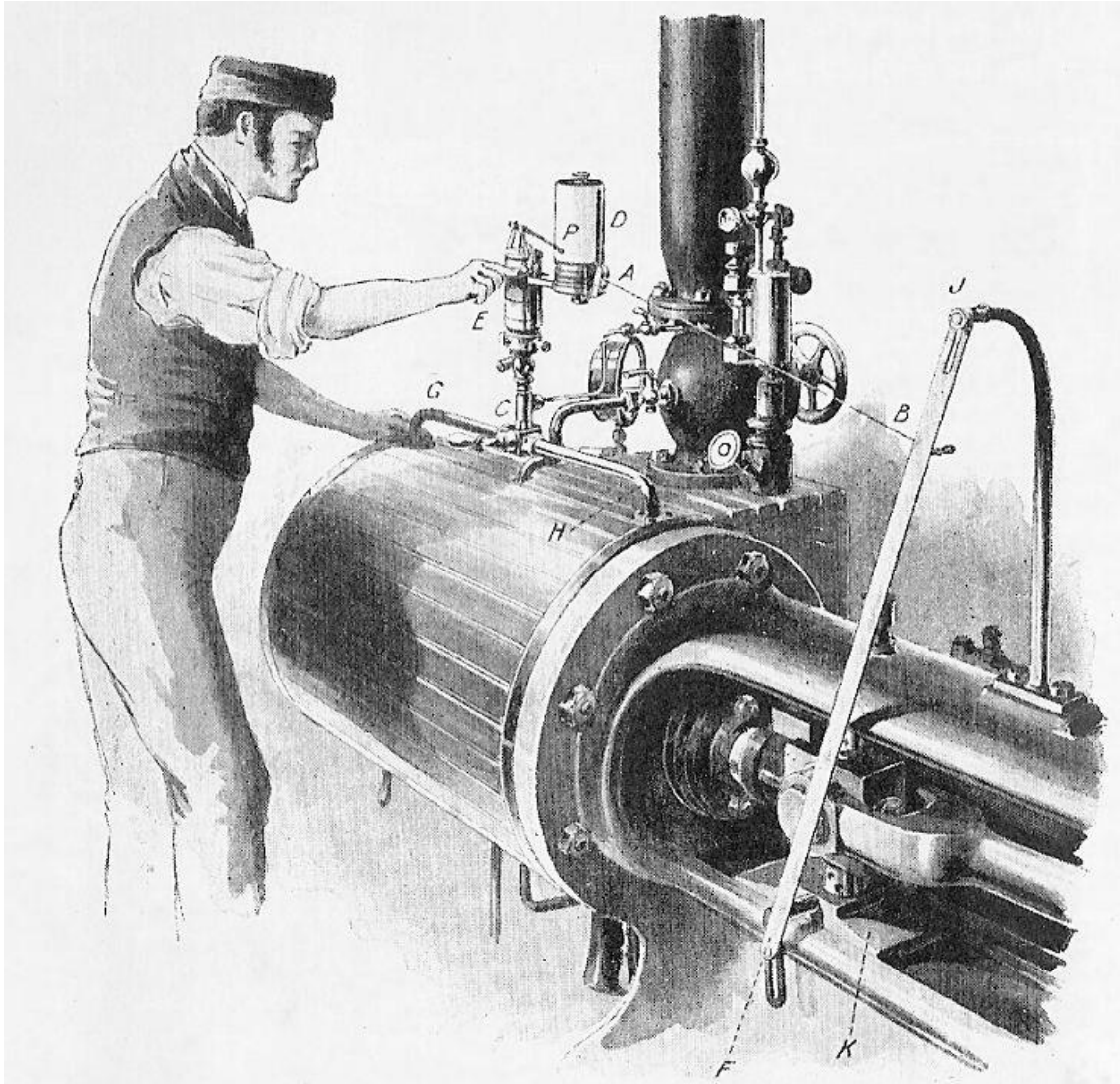


Recording pen

Rotating cylinder holds the recording card.

Connection to Pressure

Using a Steam Engine Indicator

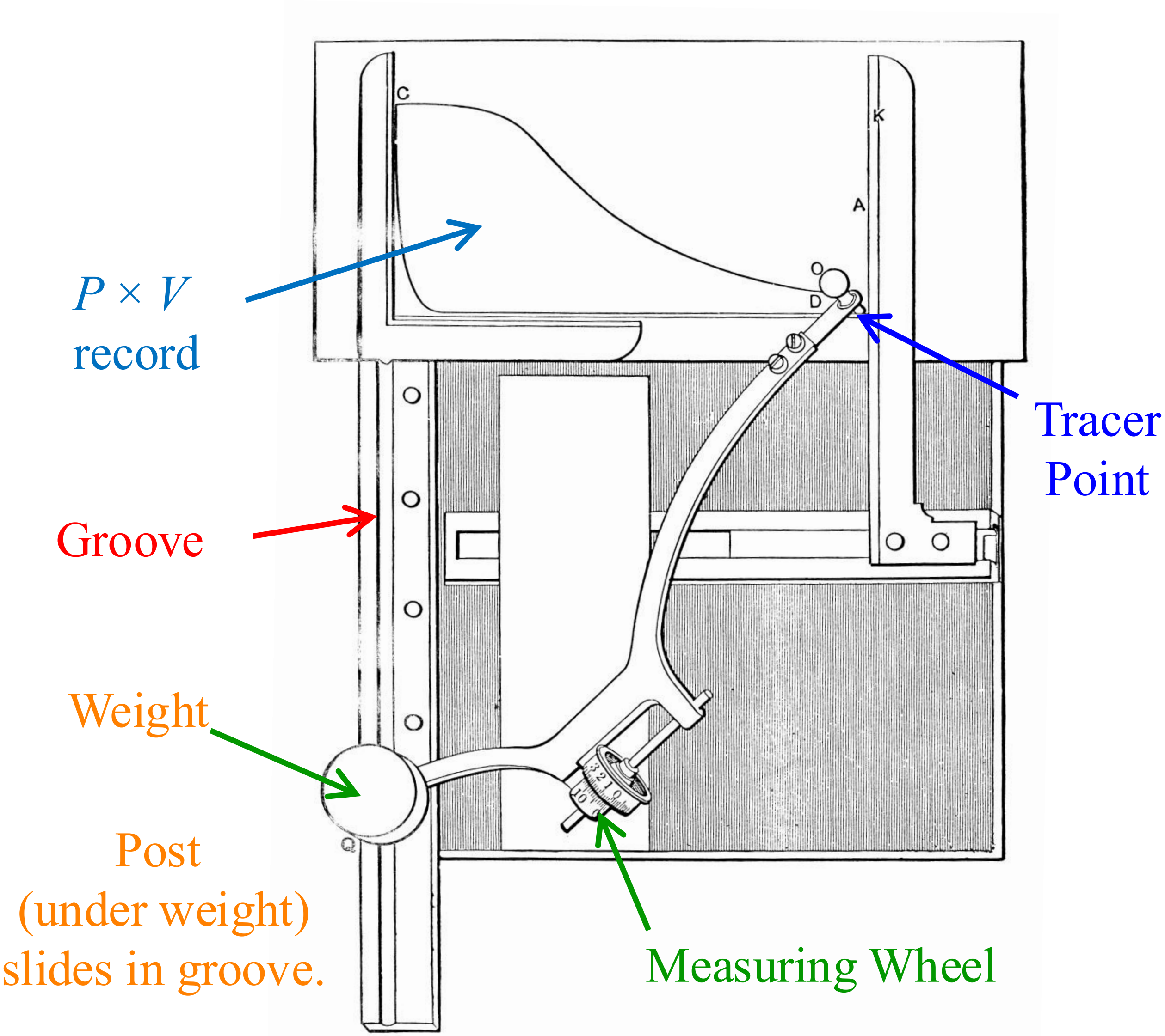


The Steam Engine by John Perry, 1902, McMillan and Co., on Intl. Slide Rule Museum website.

Planimeter Models specific for Steam Engine Testing

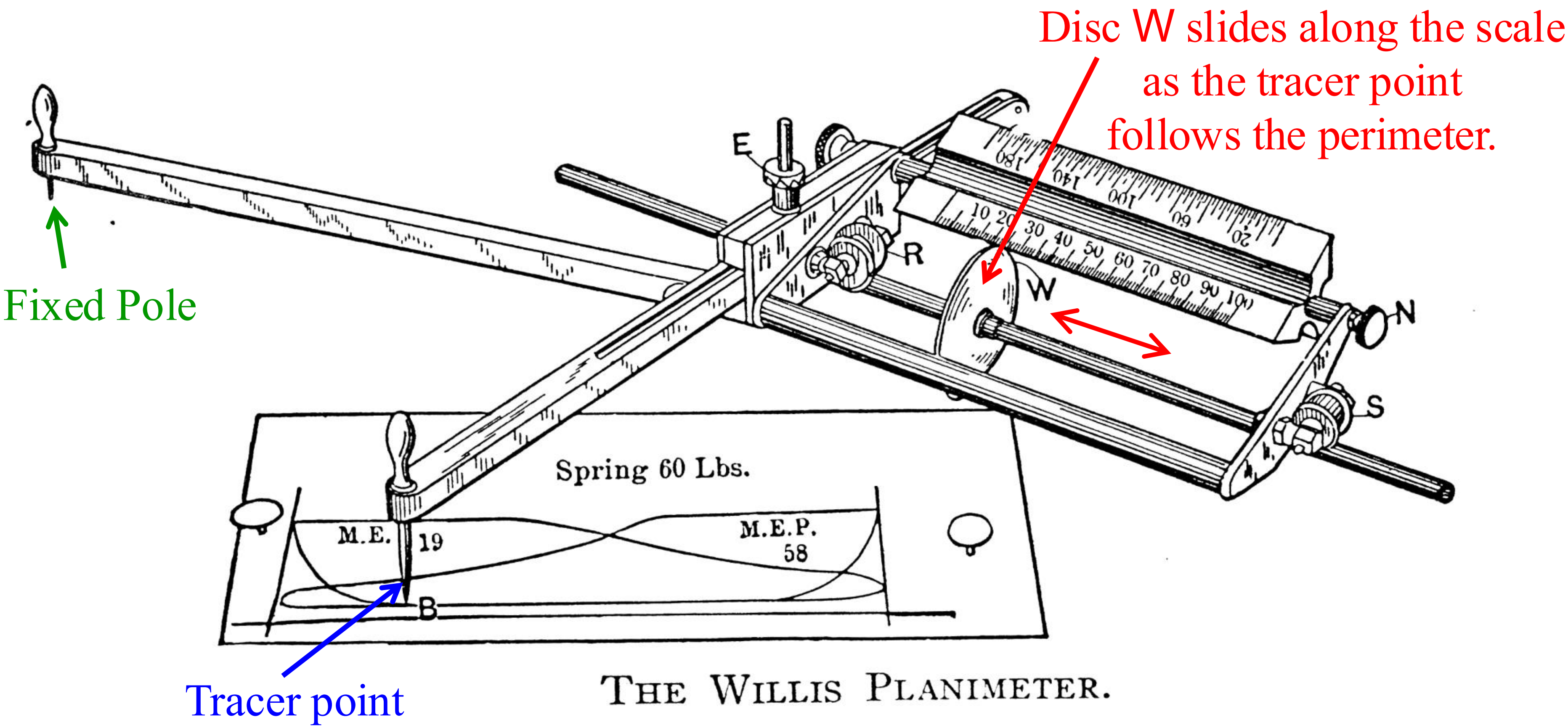
Coffin Averaging Instrument

A linear planimeter



(Carpenter & Diederichs, Experimental Engineering and Manual for Testing, Fig. 13)

The Willis Planimeter



THE WILLIS PLANIMETER.

The Willis planimeter reverses the roles of rolling and sliding.

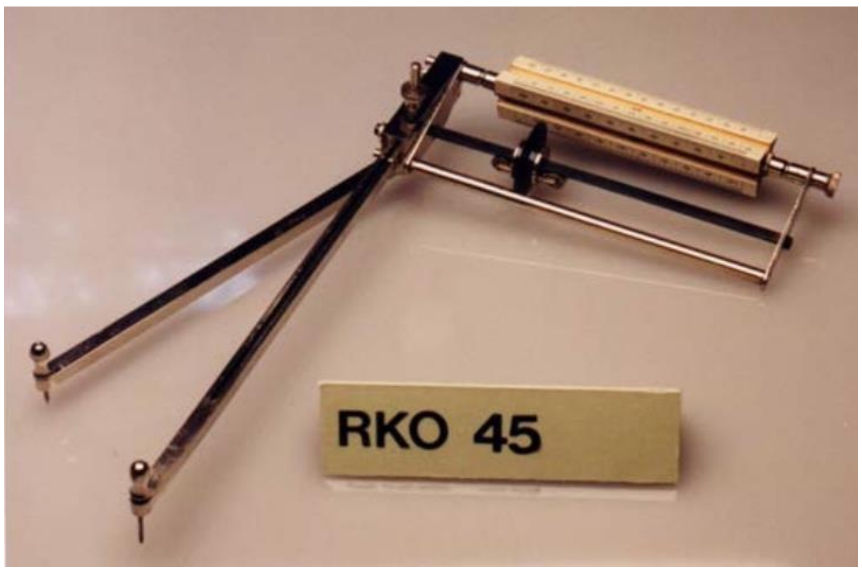
(Carpenter & Diederichs, Experimental Engineering and Manual for Testing, Fig. 15)

Willis Planimeter and Lippincott Planimeter

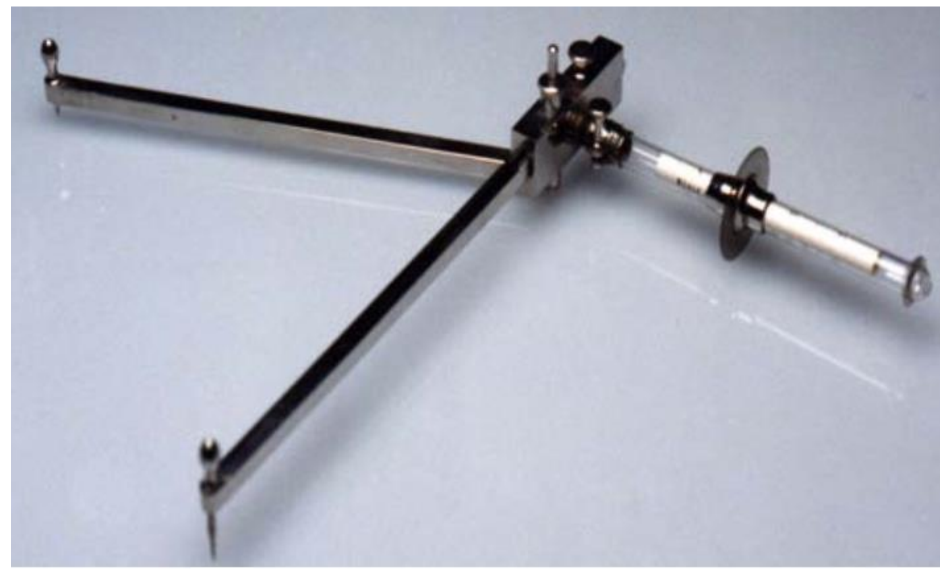
Reverse the roles of sliding and rolling:

Sliding is used for the measurement;

Rolling is neglected.



Willis

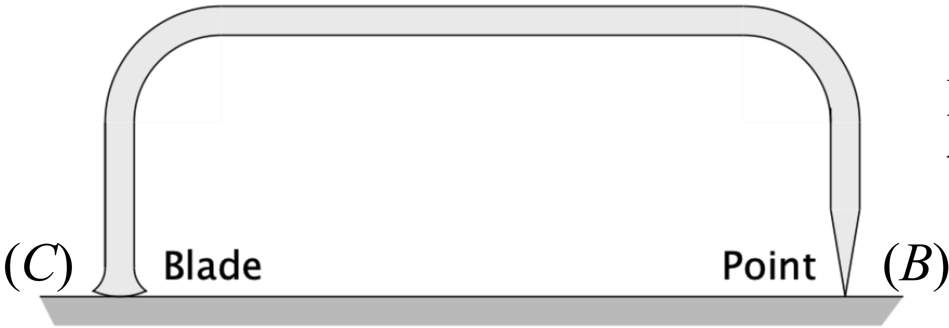


Lippincott

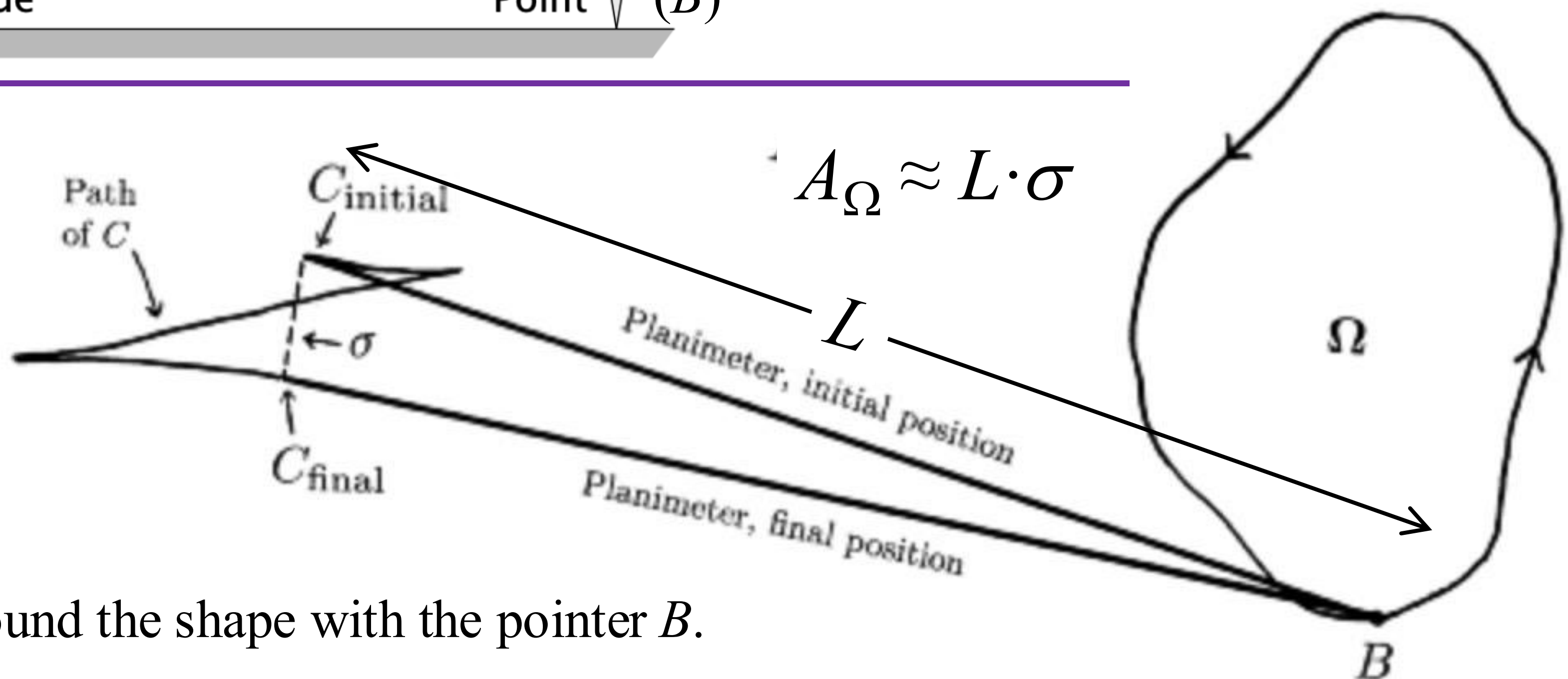
From "American Planimeters" by Bob Otnes,
J. Oughtred Society, Vol. 11, No. 2, Fall 2002, pp. 59-64

Hatchet Planimeter

Invented in ~1875 by Danish Cavalry Officer Holgar Prytz



Prytz looked for a very cheap alternative to the well-known Amsler planimeter.

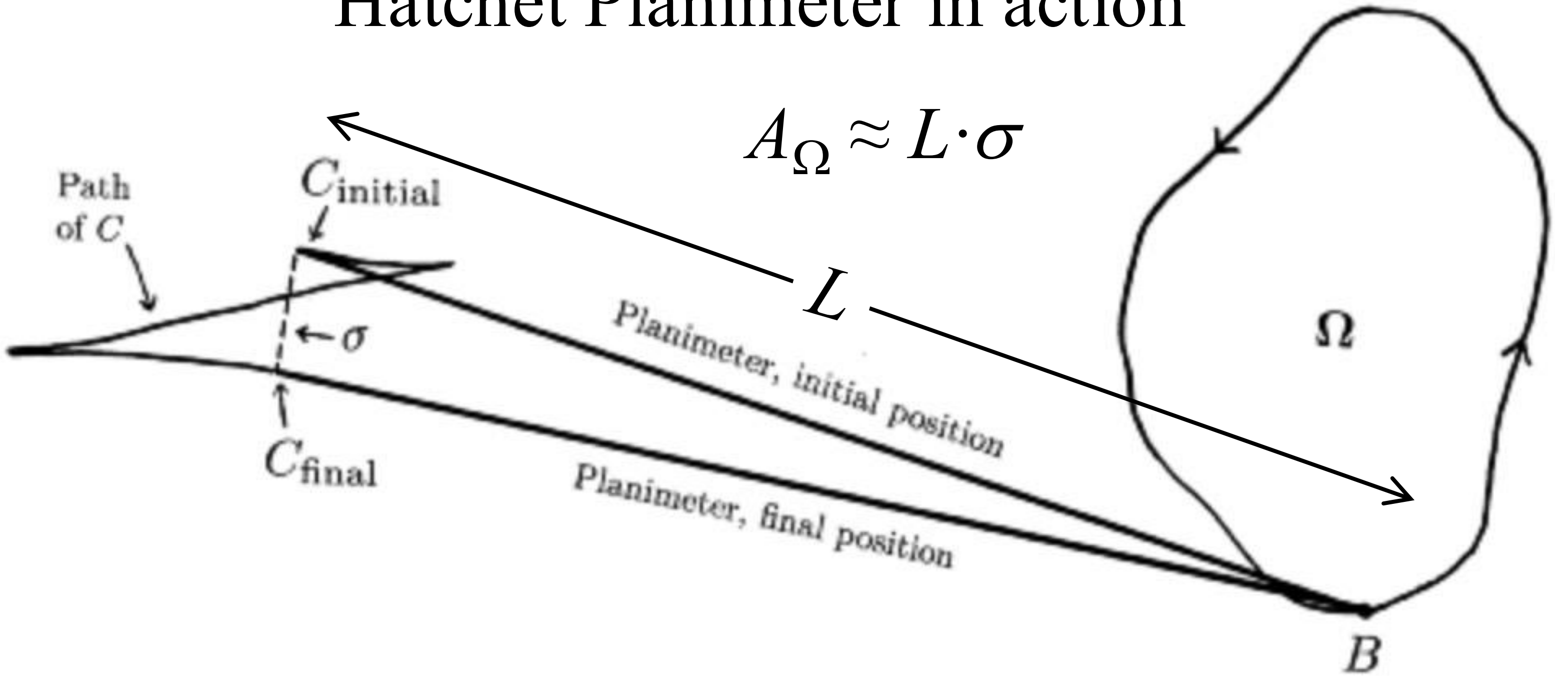


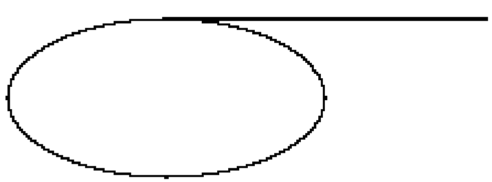
- Trace around the shape with the pointer B.
- Let the blade (C) follow whatever path it chooses.
- Measure the distance moved by the blade during the circuit (σ).
- Compute the area as $A \approx L\cdot\sigma$

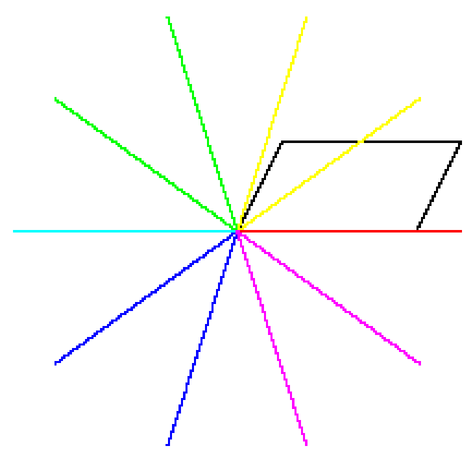
Circuit sketch from Robert Foote, The Geometry of the Prytz Planimeter,

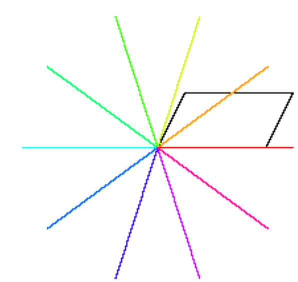
Hatchet Planimeter in action

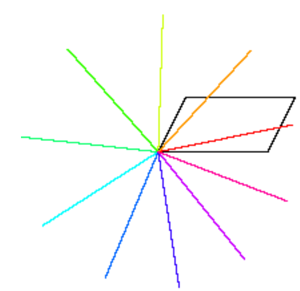
$$A_{\Omega} \approx L \cdot \sigma$$









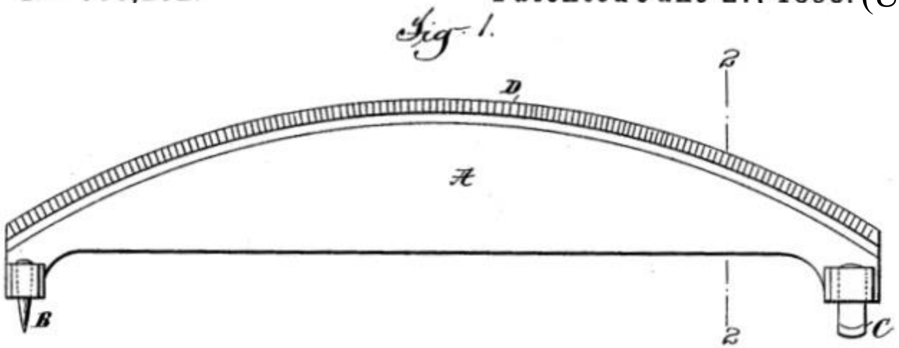


Animations from Robert Foote, The Geometry of the Prytz Planimeter,

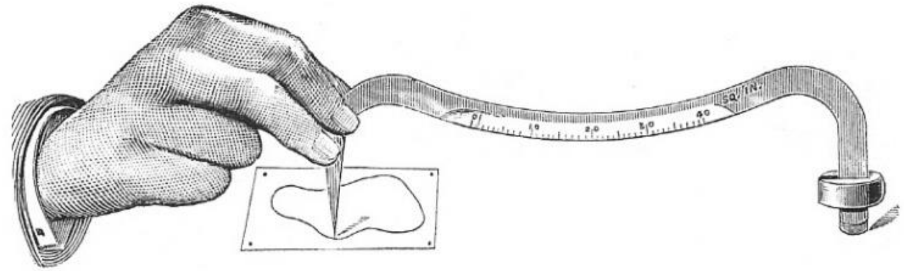
Variety of Hatchet Planimeter Designs

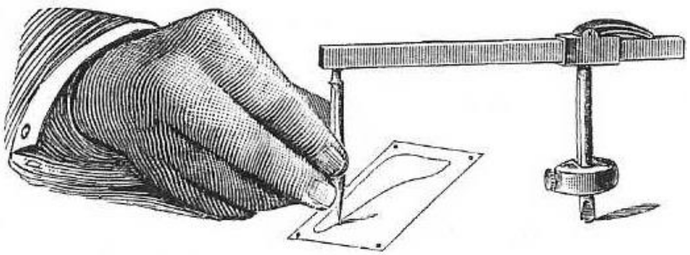
J. GOODMAN.
PLANIMETER.

No. 500,202. Patented June 27. 1893. (U.S.A.)



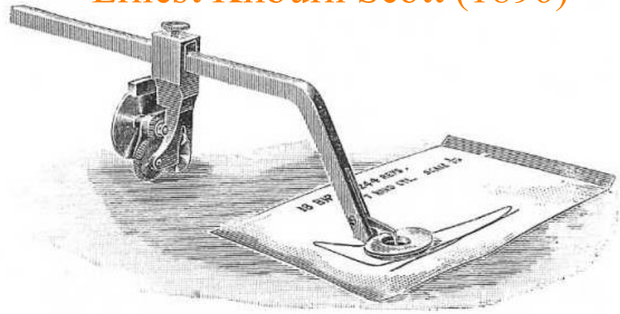
John Goodman (1891)

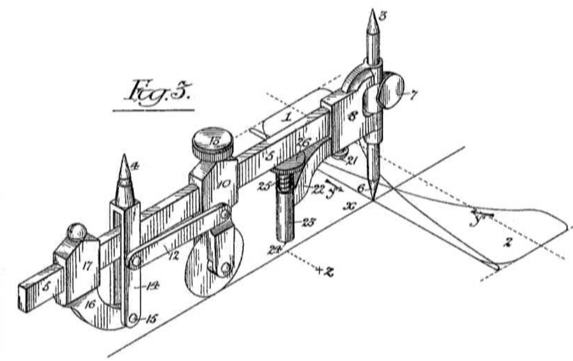




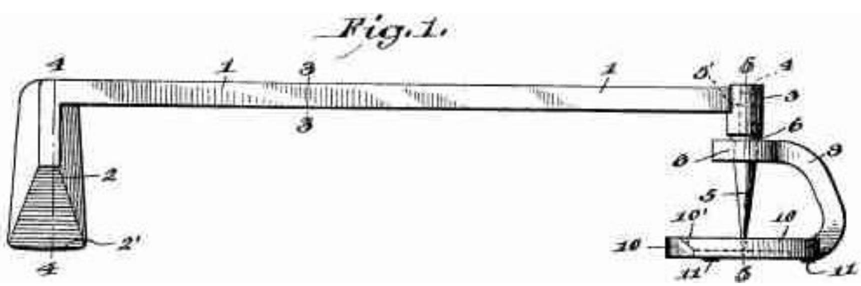
G. Coradi (1895)

Ernest Kilburn Scott (1896)



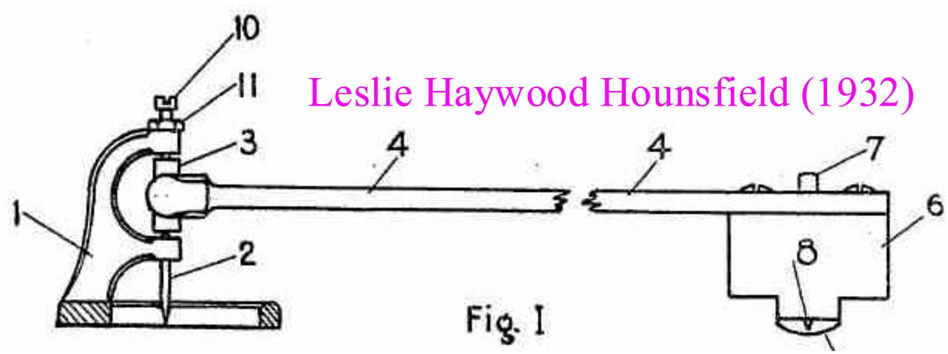


Rudolf Schierbeck (1904)



Luis Larrazabal y Fernandez (1926)

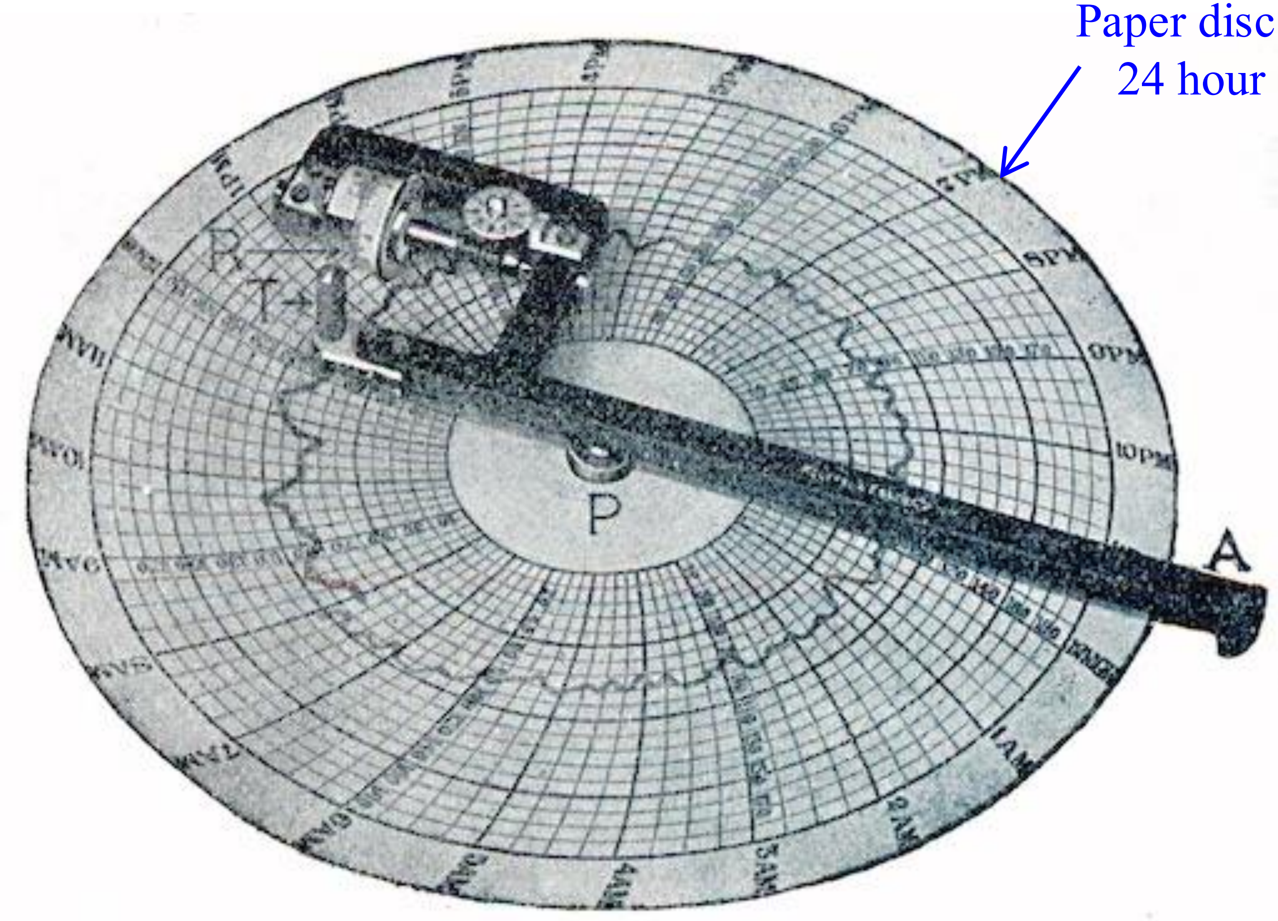
Leslie Haywood Hounsfield (1932)



Radial Planimeters

To measure the average distance from the center (pole, P) on circular charts.

Paper disc printed for 24 hour duration.



A knob at pole P fits into a groove in the underside of tracer arm A, so that the tracer point T moves easily towards or away from the pole as it makes its circuit.

(sketch taken from the Keuffel & Esser catalog, 36th Edition, 1921, p. 252; https://archive.org/details/CatalogueOfKeuffelAndEsserCo36Edition1921/page/n289/mode/2up)

Square-Root Planimeters and Flow Meters

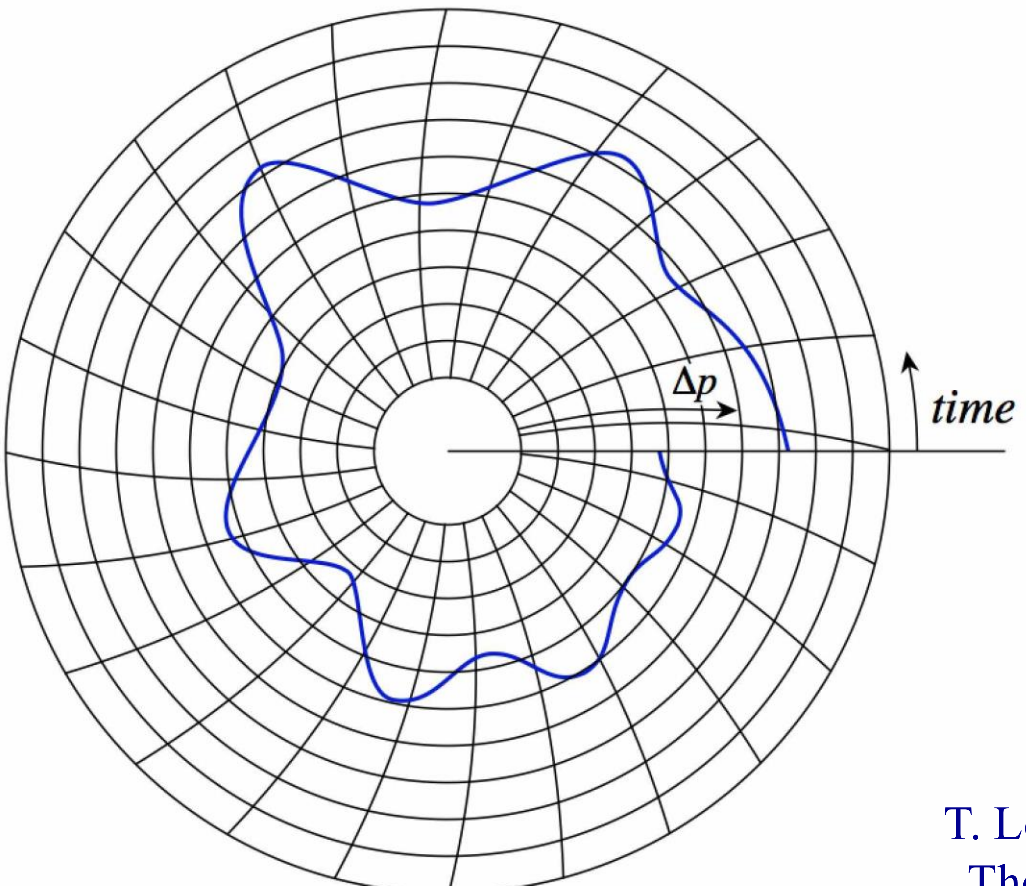
Flow meters often measure the pressure difference ($\Delta p = p_1 - p_2$) across an orifice.

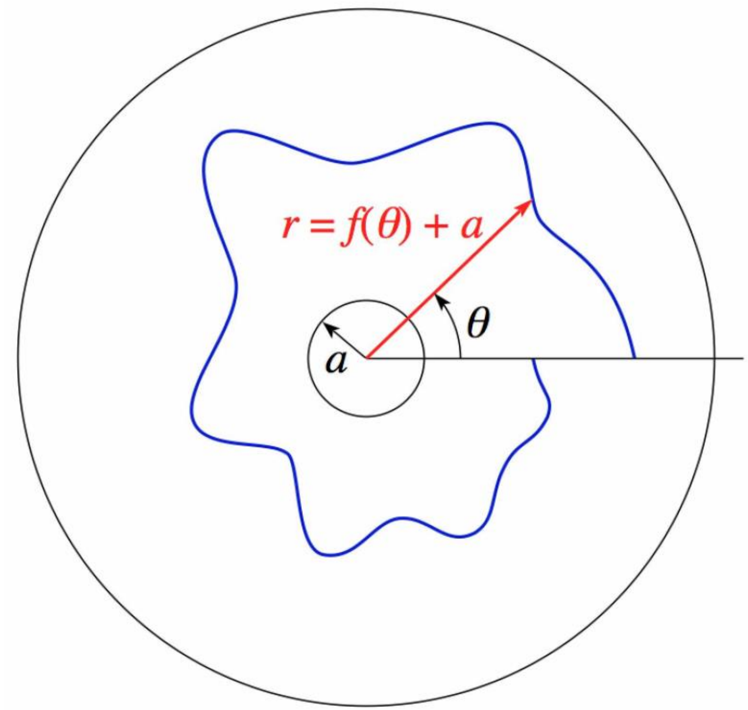
Volumetric flow rate (m^3/s): $q_v = CA_2\sqrt{2(p_1 - p_2)/\rho}$

C = orifice flow coefficient, dimensionless.

A_2 = cross-sectional area of the orifice hole, m^2.

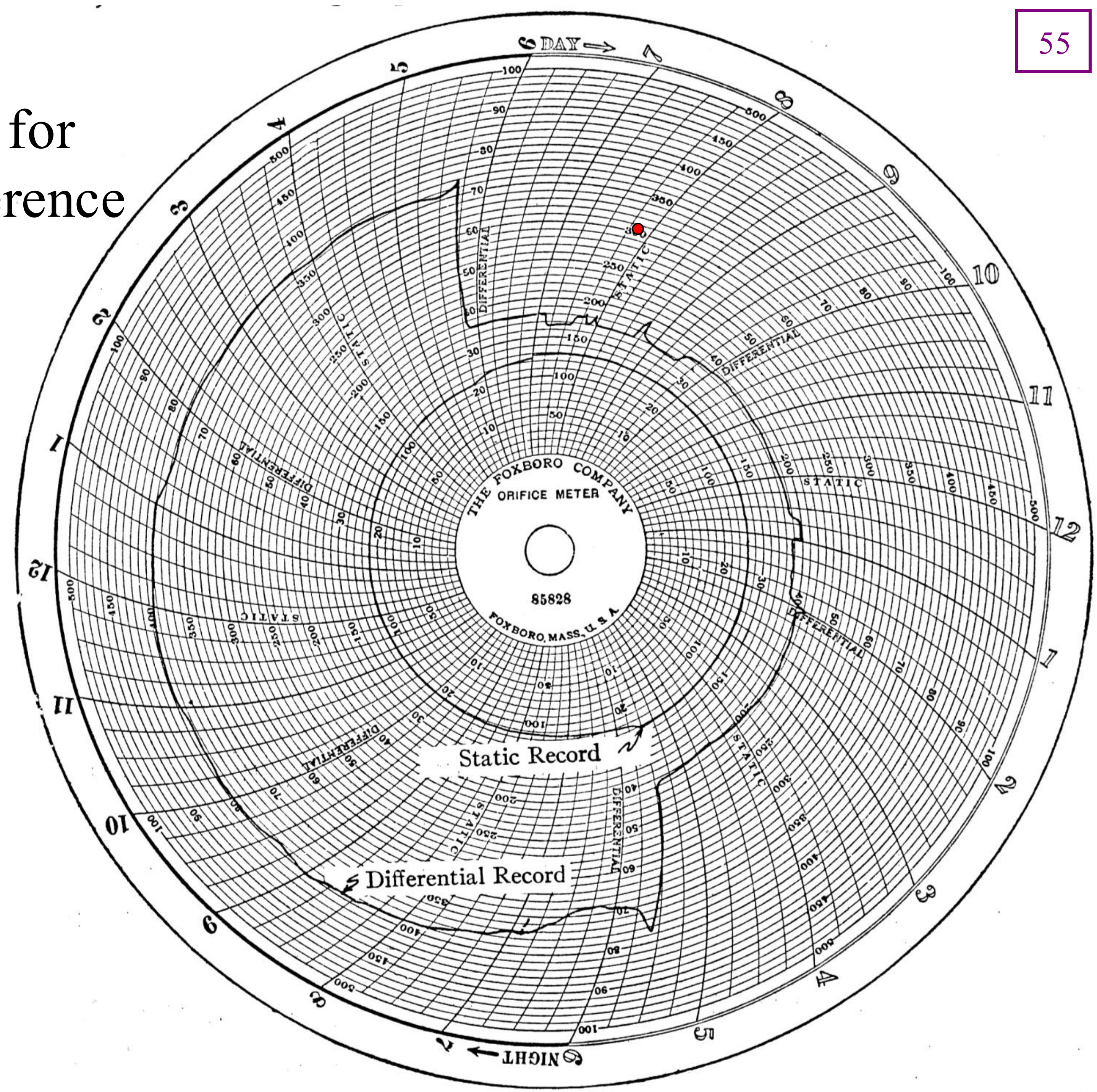
ρ = density, kg/m^3.





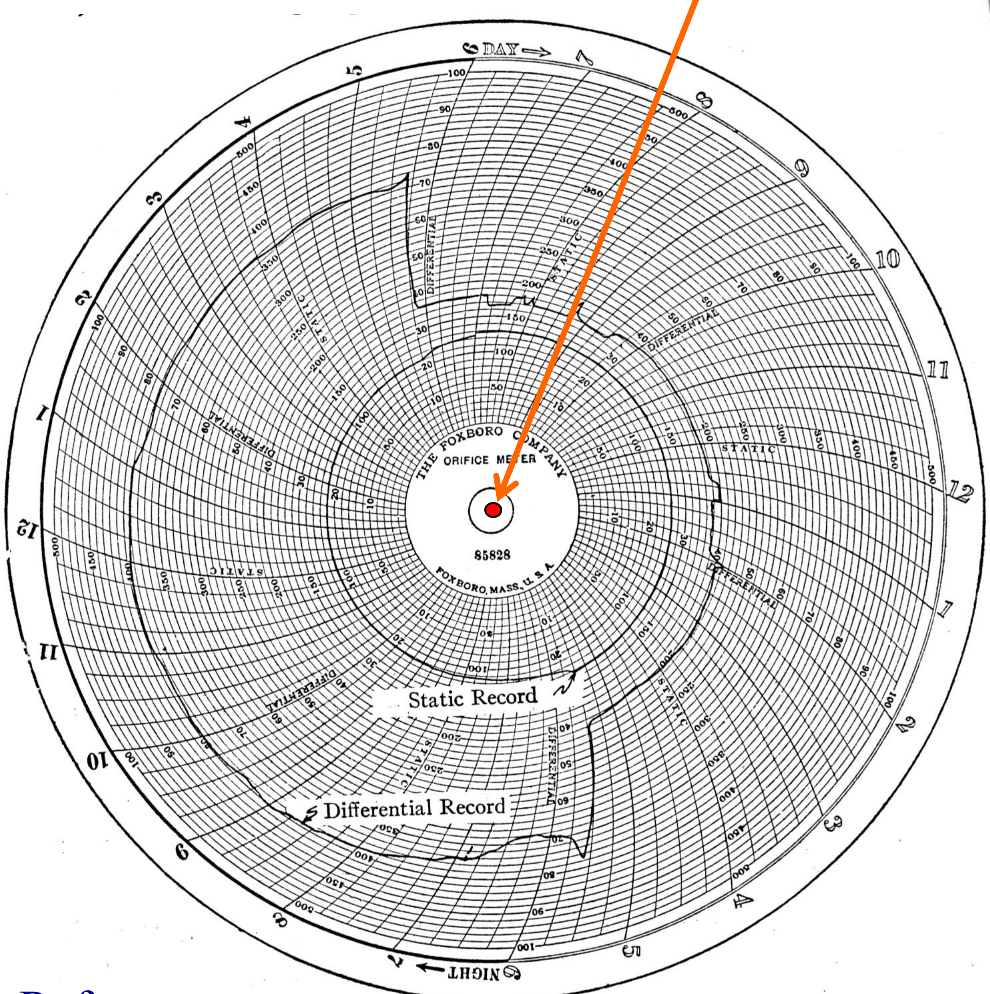
T. Leise (2007). As the planimeter's wheel rolls, The College Mathematics Journal 38(1):24-31.

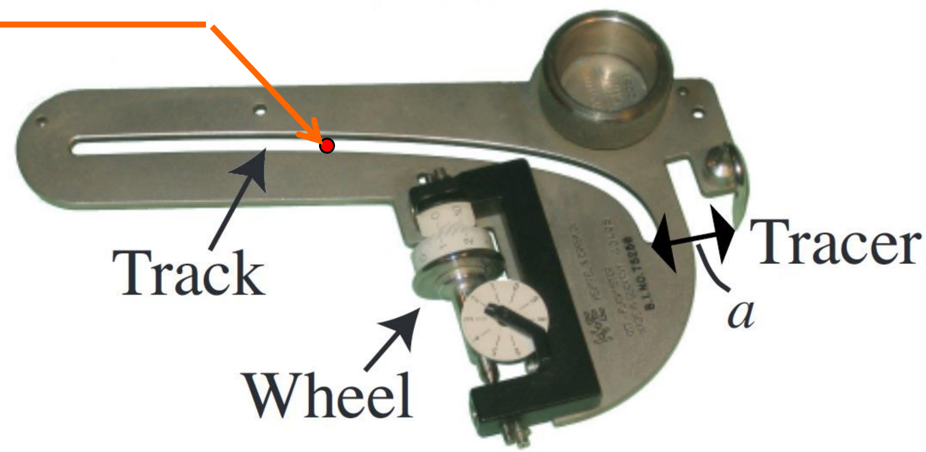
Daily Chart for Pressure Difference Record

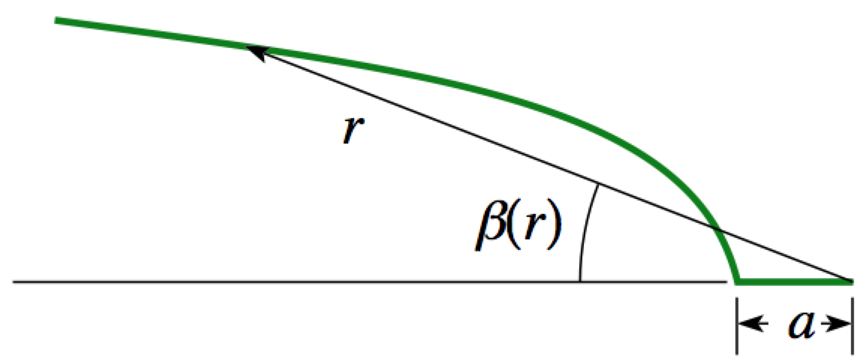


Reference:
book on flow meter engineering

Daily Chart for Pressure Difference Record



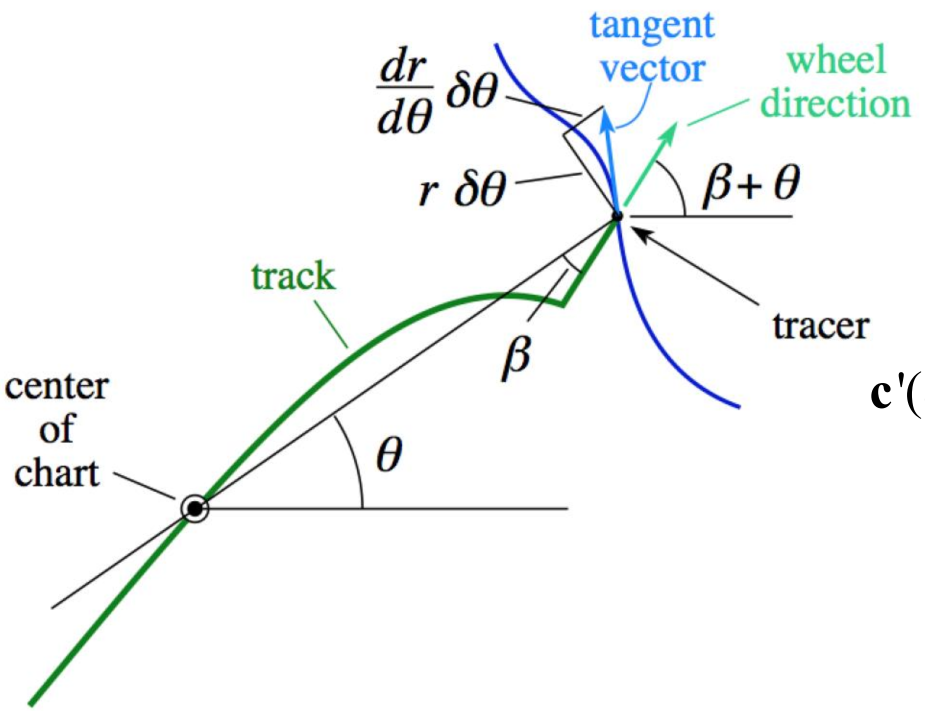




Reference:
book on flow meter engineering

Following Tanya Leise, *As the Planimeter's Wheel Turns…*

Operation on a Circular Chart



Use chevron brackets notation for vectors:

$$\langle x, y\rangle \square x\mathbf{i} + y\mathbf{j}$$

$$\mathbf{c}(\theta) = \left(f(\theta)+a\right)\langle\cos\theta, \sin\theta\rangle$$

$$\mathbf{c}'(\theta) = \left(f(\theta)+a\right)\langle -\sin\theta, \cos\theta\rangle + \frac{df}{d\theta}\langle\cos\theta, \sin\theta\rangle$$

unit vector for wheel rolling direction:

$$\mathbf{w}(\theta) = \langle\cos(\theta+\beta), \sin(\theta+\beta)\rangle$$

The increment recorded by the wheel rolling during an increment of tracing along the curve:

$$\mathbf{w}(\theta)\cdot\mathbf{c}'(\theta)d\theta = \langle\cos(\theta+\beta), \sin(\theta+\beta)\rangle\cdot\left[\left(f(\theta)+a\right)\langle -\sin\theta, \cos\theta\rangle + \frac{df}{d\theta}\langle\cos\theta, \sin\theta\rangle\right]d\theta$$

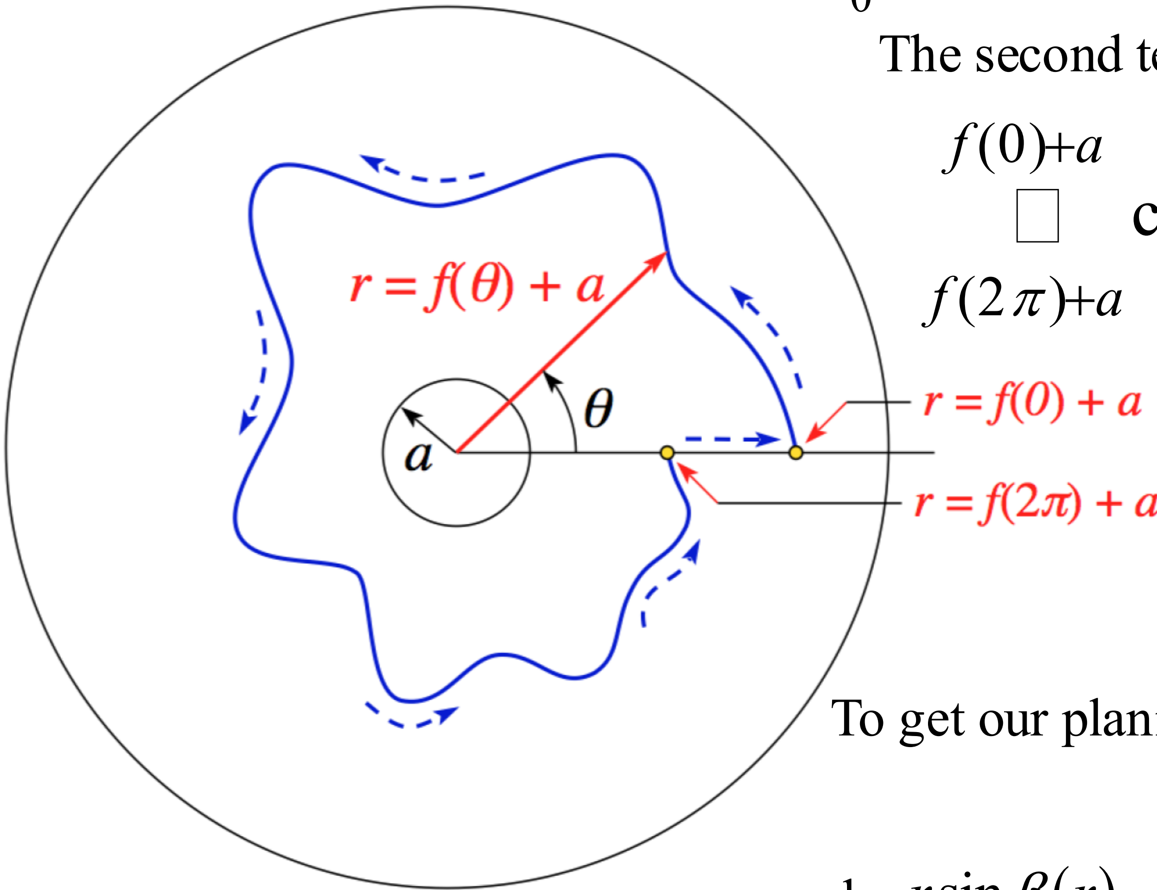
$$\mathbf{w}(\theta)\cdot\mathbf{c}'(\theta)d\theta = \left[\left(f(\theta)+a\right)\sin\beta + \frac{df}{d\theta}\cos\beta\right]d\theta$$

Following Tanya Leise, *As the Planimeter's Wheel Turns…*

Average Flow Calculation

Total Wheel Roll: $$W = \int_0^{2\pi} \mathbf{w}(\theta) \cdot \mathbf{c}'(\theta)\, d\theta + \int_{f(2\pi)+a}^{f(0)+a} \cos\beta(r)\, dr$$

The first integral has two terms: $$\int_0^{2\pi} \left(f(\theta)+a\right)\sin\beta\, d\theta + \int_0^{2\pi} \frac{df}{d\theta}\cos\beta\, d\theta$$



The second term cancels the second integral:

$$\int_{f(2\pi)+a}^{f(0)+a} \cos\beta(r)\, dr = -\int_0^{2\pi} \frac{df}{d\theta}\cos\beta\, d\theta$$

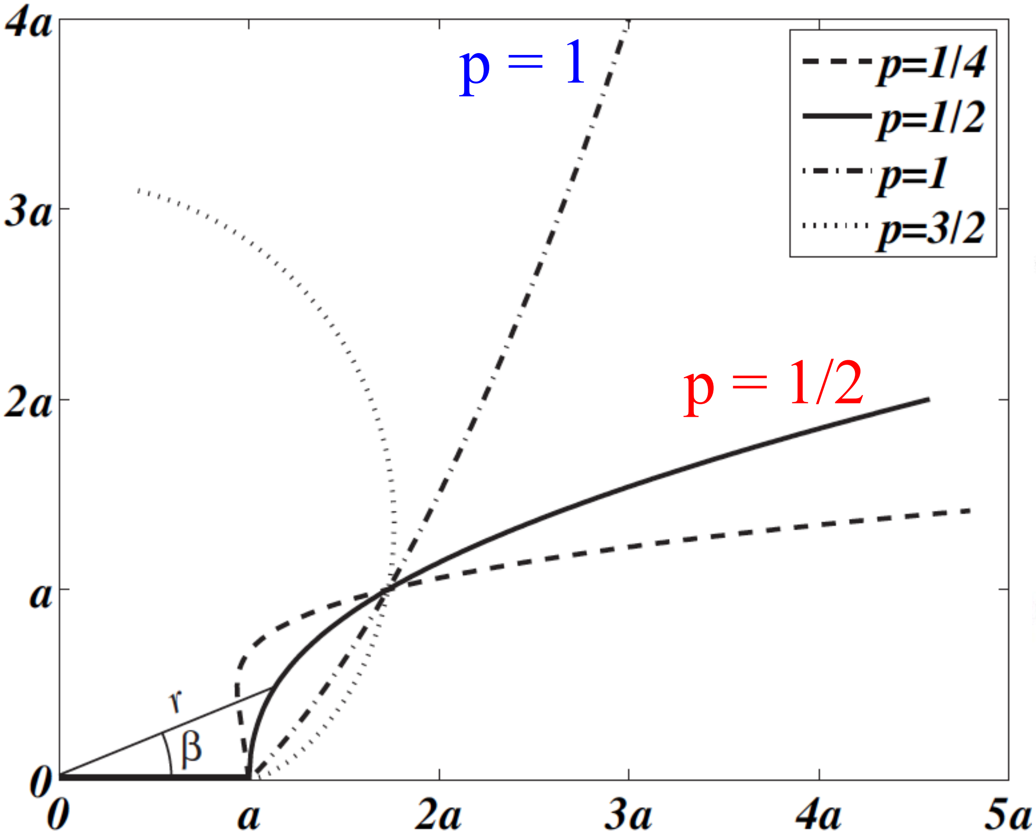
Leaving:

$$W = \int_0^{2\pi} \left(f(\theta)+a\right)\sin\beta\, d\theta$$

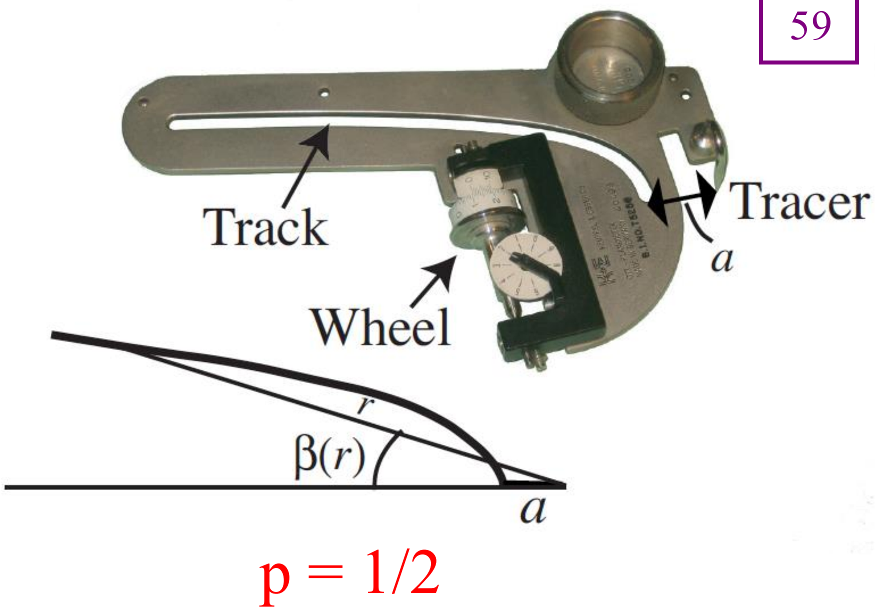
To get our planimeter to calculate $\int_0^{2\pi} \sqrt{f(\theta)}\, d\theta$

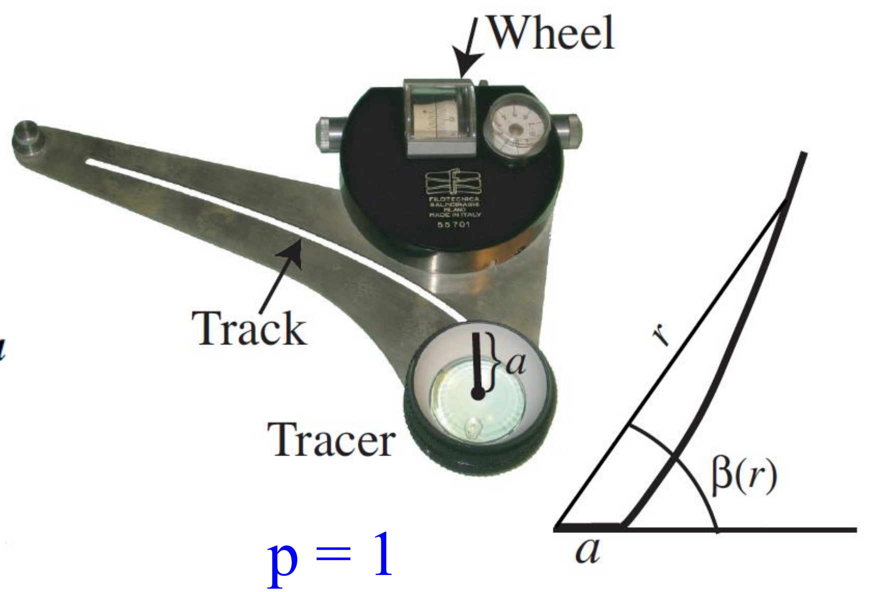
we need: $r\sin\beta(r) = \sqrt{r-a}$; $\beta(r) = \arcsin\left(\frac{\sqrt{r-a}}{r}\right)$

Radial Track Planimeters



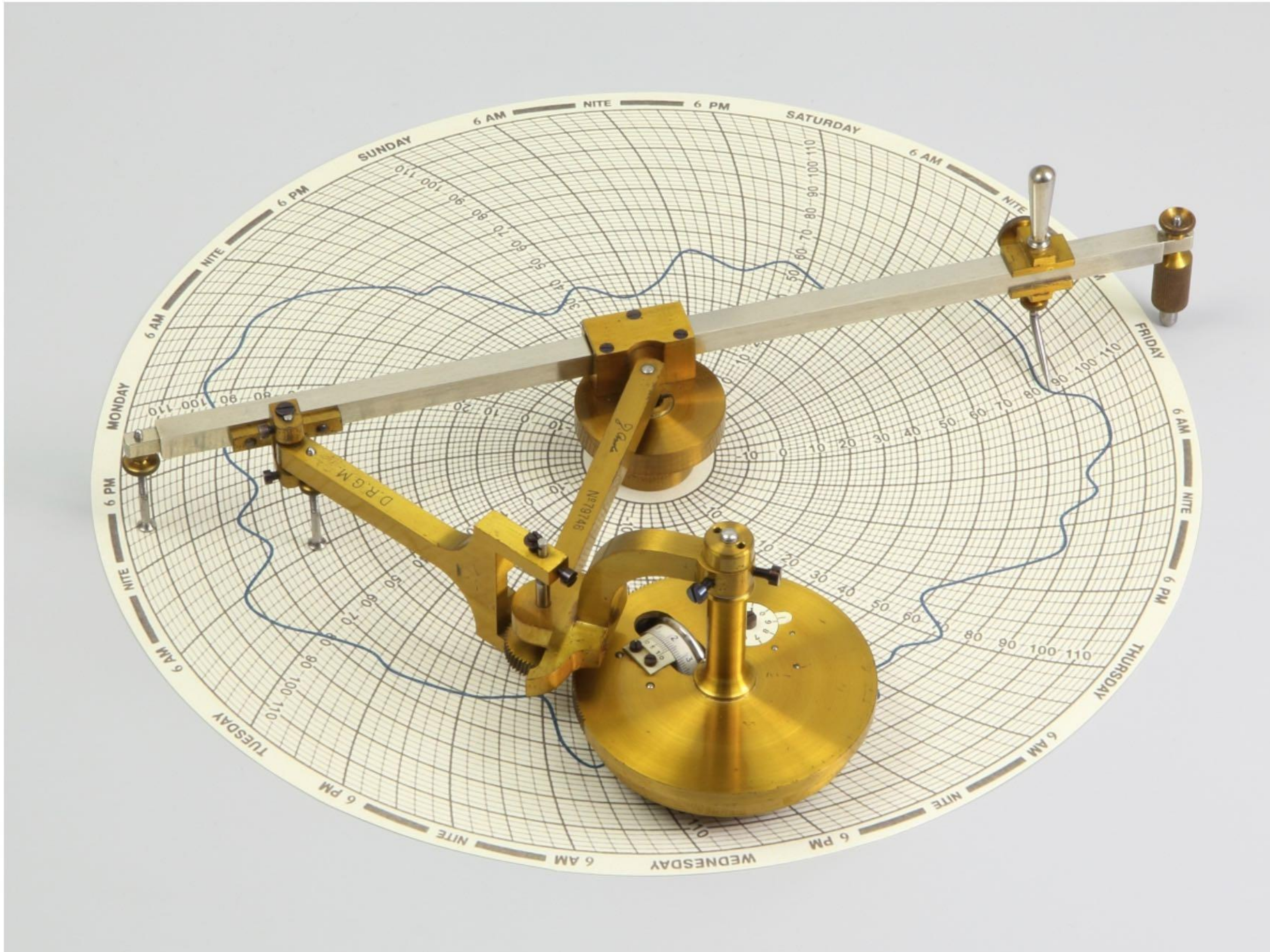
$$\beta(r) = \arcsin\left(\frac{(r-a)^p}{r}\right)$$





From Tanya Leise, *As the Planimeter's Wheel Turns…*

Amsler’s Radial Square-Root Planimeter



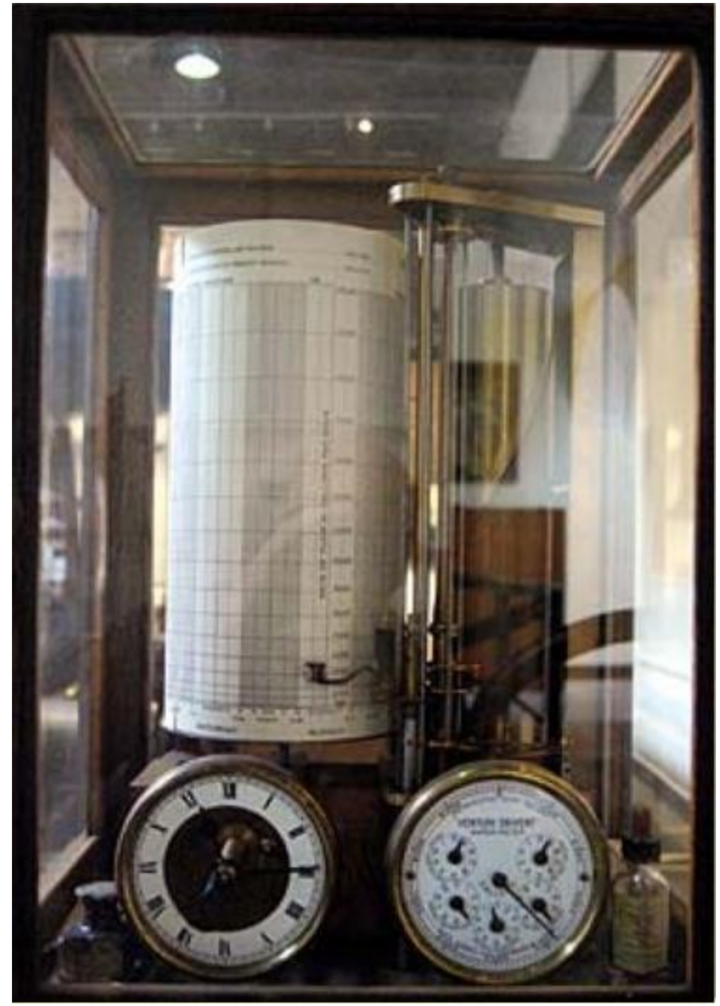
(An adequate description of how this gadget works is not yet available.)

Photograph from: http://www.mathinstruments.ch/en/gallery/inv-3013.html

Linear Square-Root Planimeter

by Geo Kent Ltd. London & Luton.

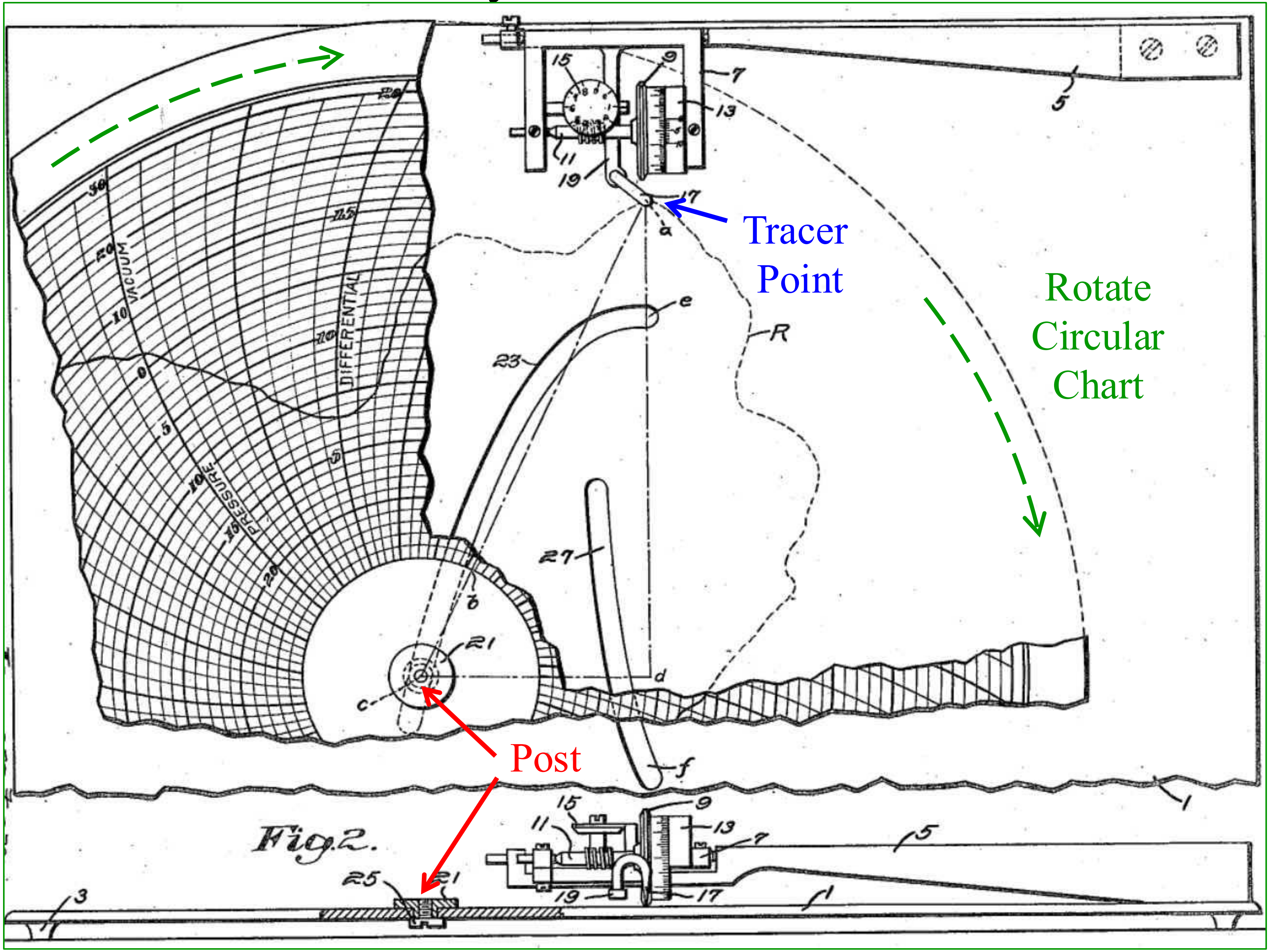
Planimeter intended to run along a rail that is not shown.



A Geo Kent 'Venturi' water flow rate meter in Kew Bridge Steam Museum.

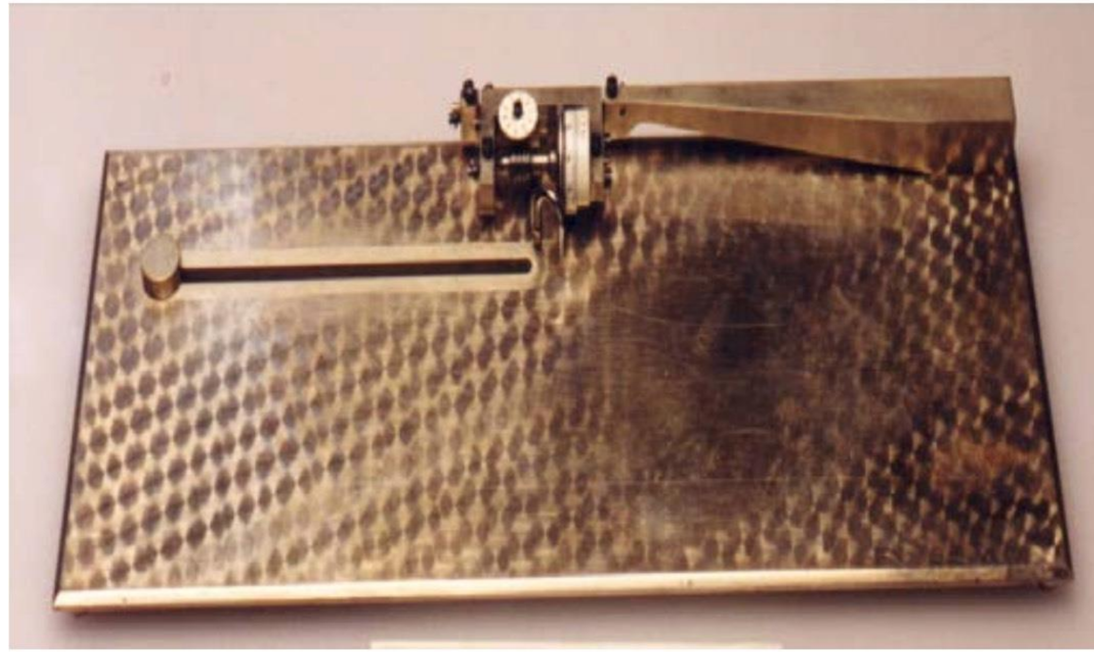
Planimeter and flow meter as shown on website http://www.mathsinstruments.me.uk/page38.html

Foxboro-Style Radial Planimeter



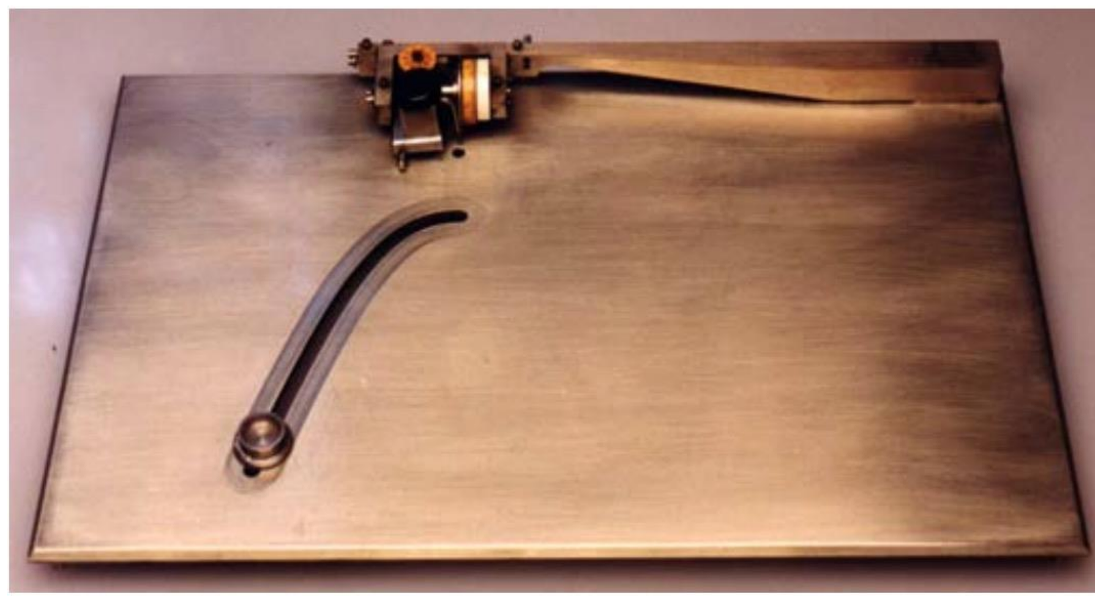
(Figures from U.S. Patent 1,650,490, Filed Sept. 13, 1926, by W. C. Brown and L. K. Spink)

Linear Scale



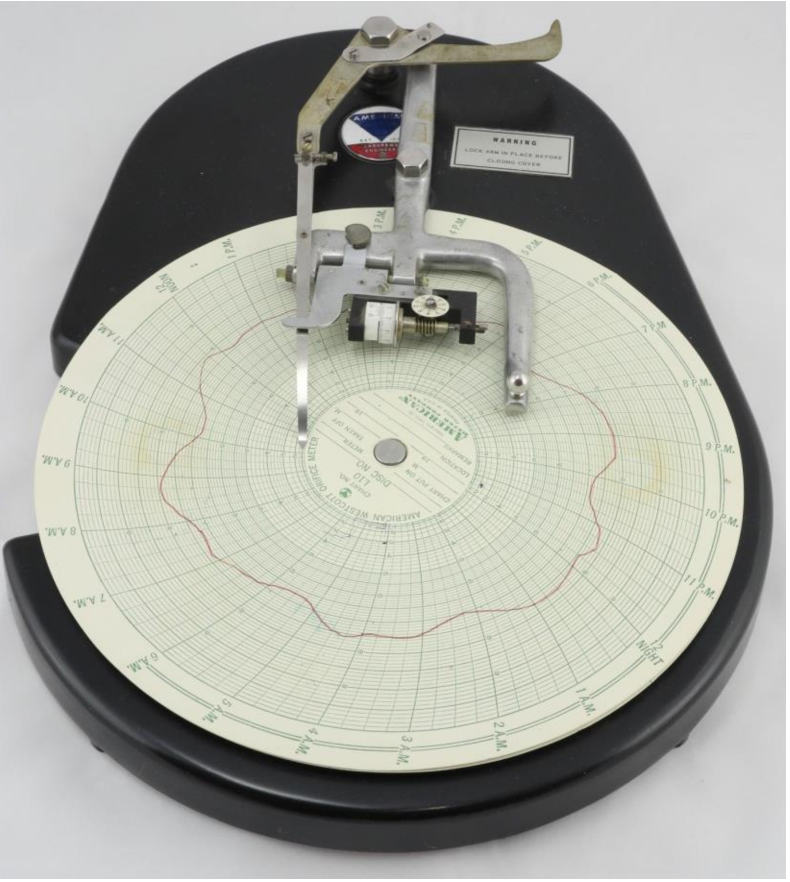
Foxboro Radial Planimeters

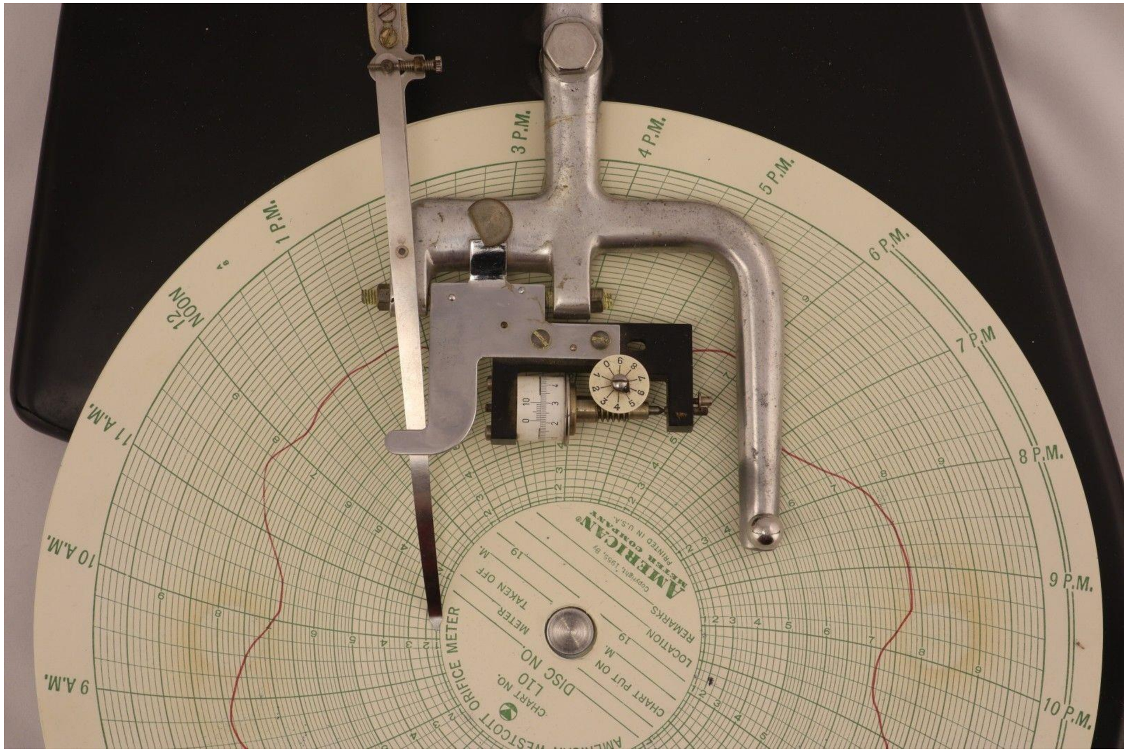
Square-Root Scale



From "American Planimeters" by Bob Otnes, J. Oughtred Society, Vol. 11, No. 2, Fall 2002, pp. 59-64

American Meter Co. - Square Root Planimeter





A description of this instrument is given in “American Planimeters” by Bob Otnes, J. Oughtred Society, Vol. 11, No. 2, Fall 2002, pp. 59-64

(photographs from eBay listing, Nov. 2018)

Summary for Planimeters

- Early Planimeter designs, such as Cone and Disc Mechanisms.
- Jacob Amsler's invention of the rolling + sliding wheel.
- Operation of the Linear Planimeter.
- Mathematics of the Polar Planimeter.
- Steam Engine Indicators.
- Prytz (Hatchet) Planimeters.
- Flow Meters and Square Root Planimeters.

Next Lecture: Amsler Integrators and other derivative gadgets.