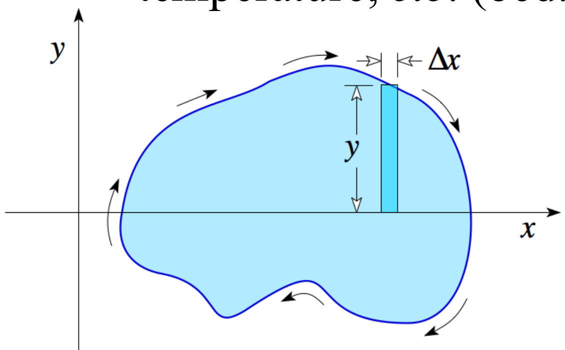
Amsler Integrators and Related Devices

The mechanical measurement of the Center of Mass, Moment of Inertia, and the like.

Integrator, *n*: One who or that which integrates; *spec*. an instrument for indicating or registering the total amount or mean value of some physical quantity, as area, temperature, *etc*. (oed.com, 2013)

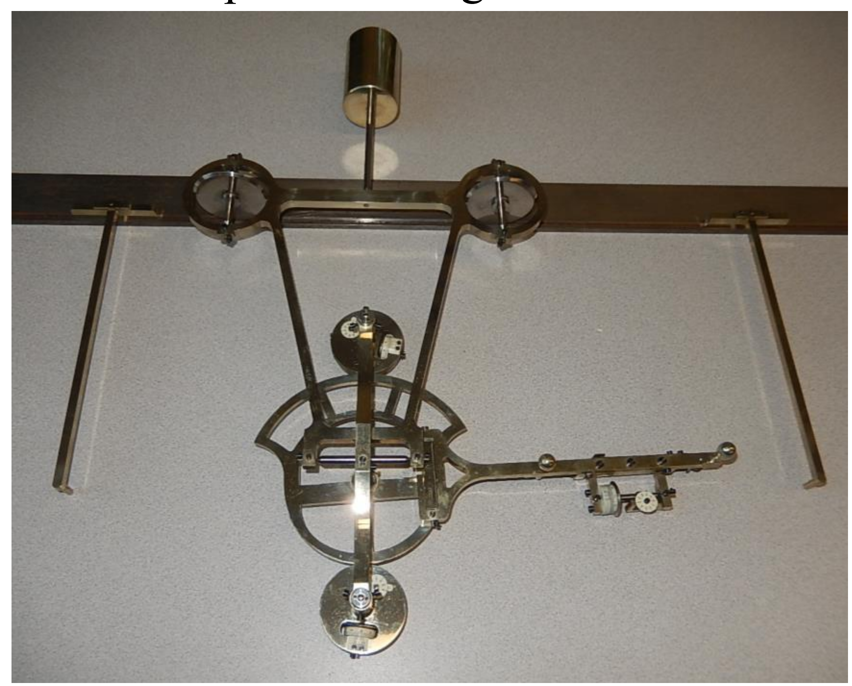


$$A = \int_A dA = \oint_{perimeter} y\,dx$$

$$Q_x = \int_A y \cdot dA = \tfrac{1}{2}\oint y^2\,dx$$

$$I_{xx} = \int_A y^2\,dA = \tfrac{1}{3}\oint y^3\,dx$$

Soon after inventing a planimeter in 1854, Jakob Amsler extended his ideas to mechanisms that measure more complicated integrals.



Computing before Electronic Calculators
Thayer School of Engineering
Osher at Dartmouth, Fall, 2024

Amsler Invention to measure these Integrals

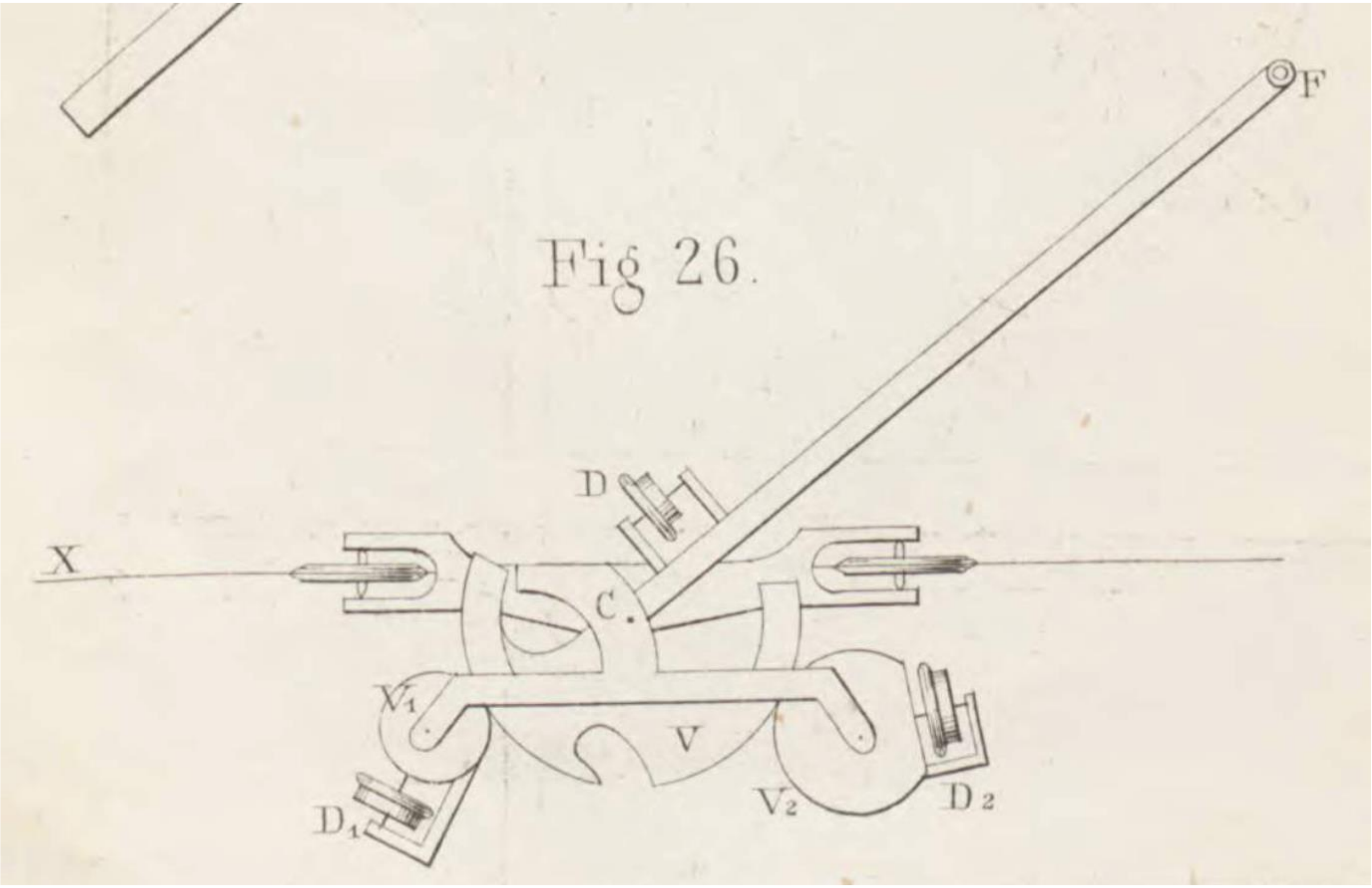


Figure from Jakob Amsler's 1856 booklet showing the basic idea of the three dial Amsler integrator.

More Complex Integrals: Moments of Area

Area:

$$A = \int_A dA = \oint_{perimeter} y\,dx$$

(Units: m^2)

First Moment of Area:

$$Q_x = \int_A y \cdot dA \; ; \quad Q_y = \int_A x \cdot dA$$

(Units: m^3)

Hibbeler* Eqn.(12-2)

("Static Moment" used to get center of mass and for calculating shear stresses in bending.)
(Used but not named in Hibbeler Section 6.1)

Second Moment of Area:

$$I_x = \int_A y^2\,dA \; ; \quad I_y = \int_A x^2\,dA$$

(Units: m^4)

Hibbeler Eqn.(6-10)

("Moment of Inertia"; used in bending calculations.)

Third Moment of Area:

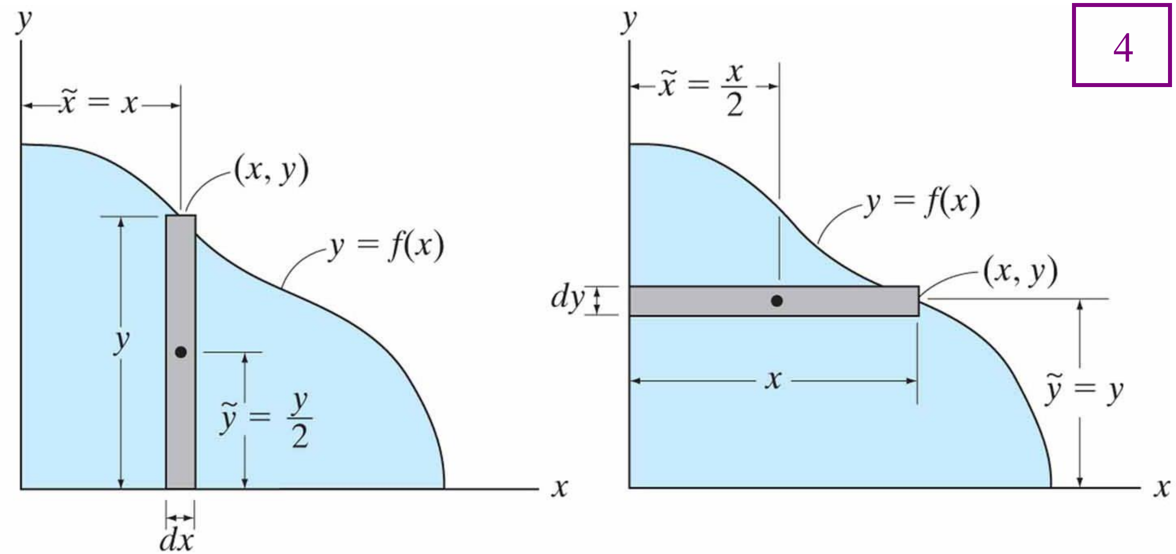
$$\int_A y^3\,dA$$

(Units: m^5)

* Text for ENGS 33: Statics and Mechanics of Materials, R.C. Hibbeler, Prentice Hall

Moments of Area

For shapes bordering the coordinate axes, defined by $y = f(x)$.



Area:

$$A = \int_A dA = \int_{x_{\min}}^{x_{\max}} y\,dx$$

(Units: m^2)

First Moment of Area:

$$Q_x = \int_A y\,dA \quad ; \quad Q_y = \int_A x\,dA$$

(Units: m^3)

(Used in bending calculations.)

$$Q_x = \frac{1}{2}\int_{x_{min}}^{x_{max}} y^2\,dx \; ; \quad Q_y = \frac{1}{2}\int_{y_{min}}^{y_{max}} x^2\,dy$$

Second Moment of Area:

$$I_x = \int_A y^2\,dA \quad ; \quad I_y = \int_A x^2\,dA$$

(Units: m^4)

("Moment of Inertia")

$$I_x = \frac{1}{3}\int_{x_{min}}^{x_{max}} y^3\,dx \; ; \quad I_y = \frac{1}{3}\int_{y_{min}}^{y_{max}} x^3\,dy$$

Third Moment of Area:

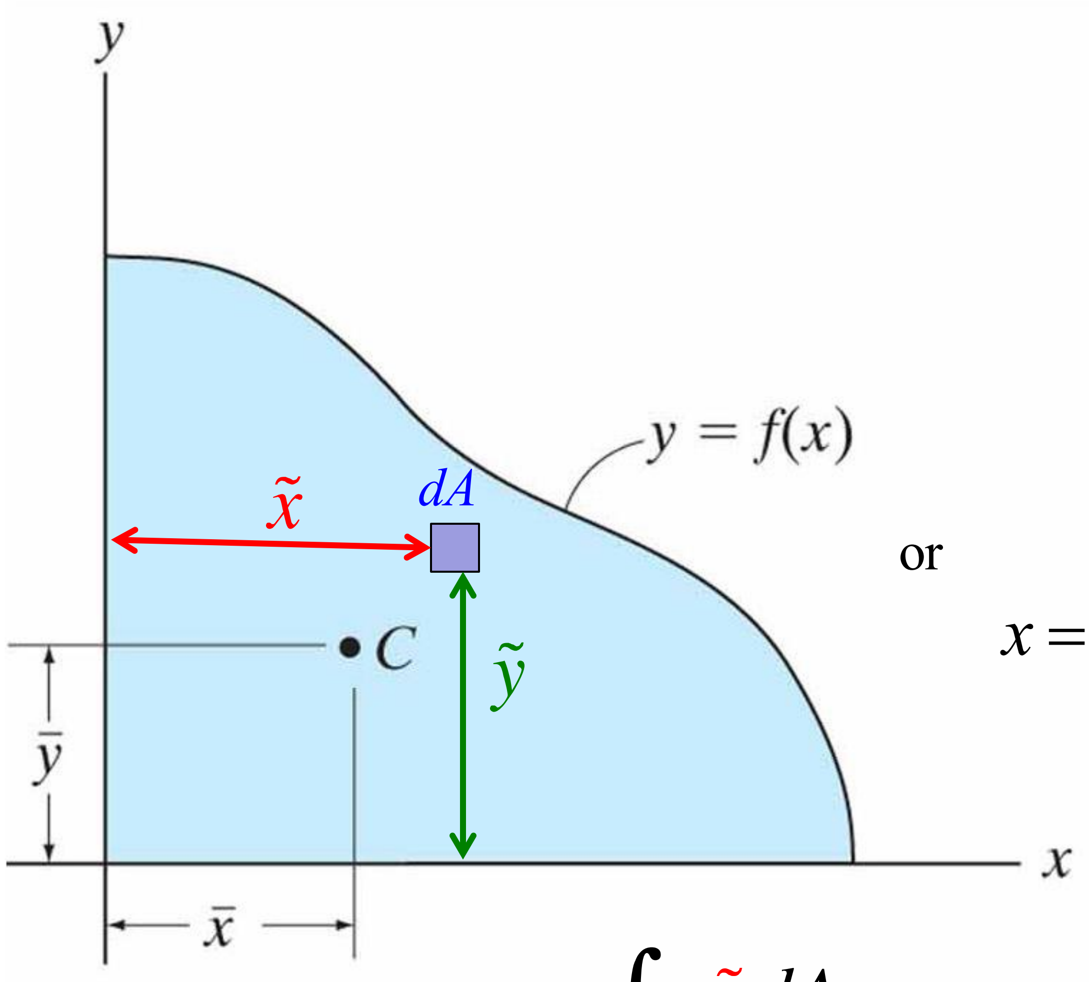
$$\int_A y^3\,dA = \frac{1}{4}\int_{x_{min}}^{x_{max}} y^4\,dx$$

(Units: m^5)

Calculation of the Centroid Location

We need two separate calculations: one for *x* location and one for *y* location.

$$\bar{y} = \frac{\int_A \tilde{y}\, dA}{\int_A dA}$$



or

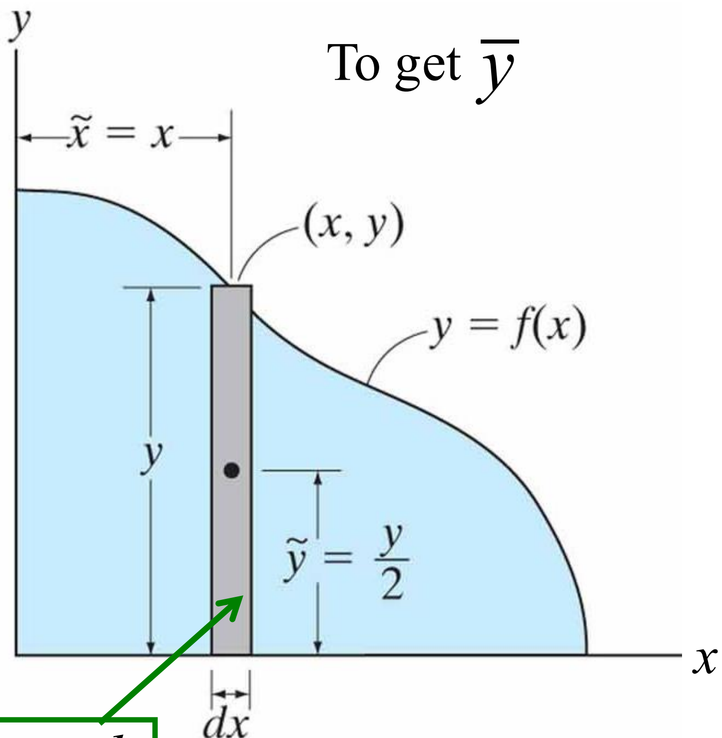
$$x = g(y)$$

$$\bar{x} = \frac{\int_A \tilde{x}\, dA}{\int_A dA}$$

(Hibbeler, Fig. 6-5a, with amendments)

Calculation of the Centroid Location

To get $\overline{y}$



$dA = y \cdot dx$

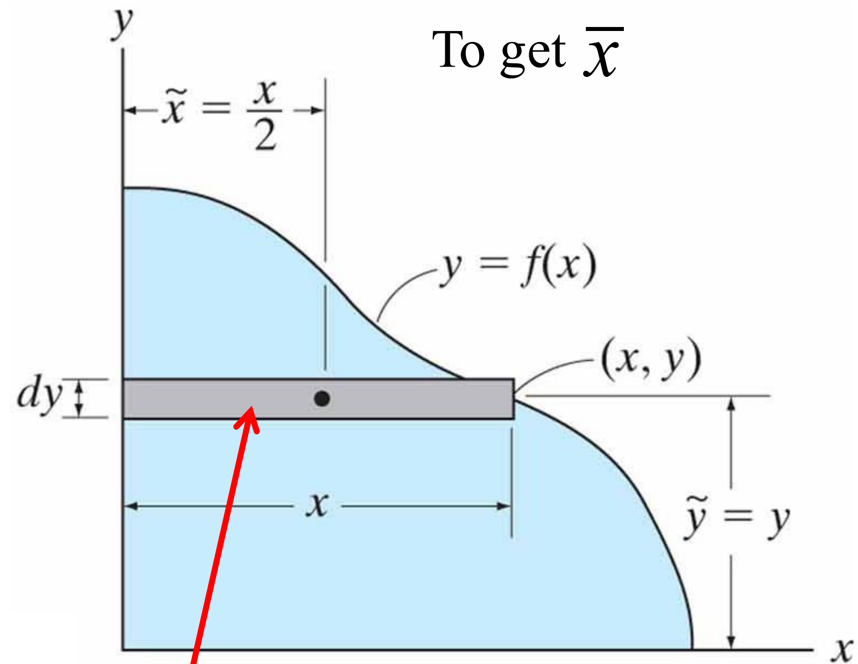
$$\overline{y} = \frac{\int_A \tilde{y}\, dA}{\int_A dA}$$

First Moment of Area:

$$Q_x = \int_A y \cdot dA = \int \frac{y}{2} y \cdot dx = \frac{1}{2} \int y^2 dx$$

$$\overline{y} = \frac{Q_x}{A}$$

To get $\overline{x}$



$dA = x \cdot dy$

$$\overline{x} = \frac{\int_A \tilde{x}\, dA}{\int_A dA}$$

$$Q_y = \int_A x \cdot dA = \int \frac{x}{2} x \cdot dy = \frac{1}{2} \int x^2 dy$$

$$\overline{x} = \frac{Q_y}{A}$$

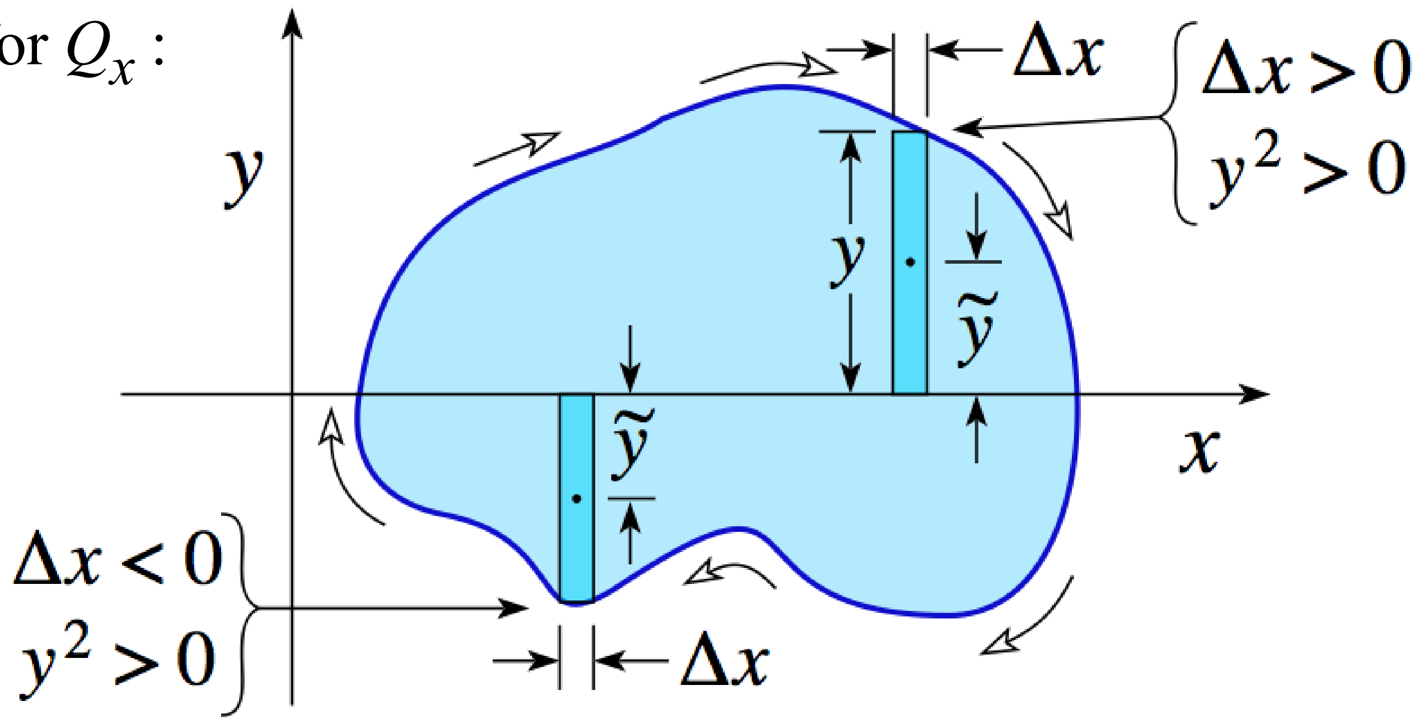
(Hibbeler, Fig. 6-5 b&c)

To find the Center of an Arbitrary Shape by Integrating around the Perimeter.

- Find the Area: A (as with a planimeter).
- Find the first moment about the horizontal: $Q_x = \frac{1}{2}\oint y^2\,dx$;
- Find the first moment about the vertical: $Q_y = \frac{1}{2}\oint x^2\,dy$

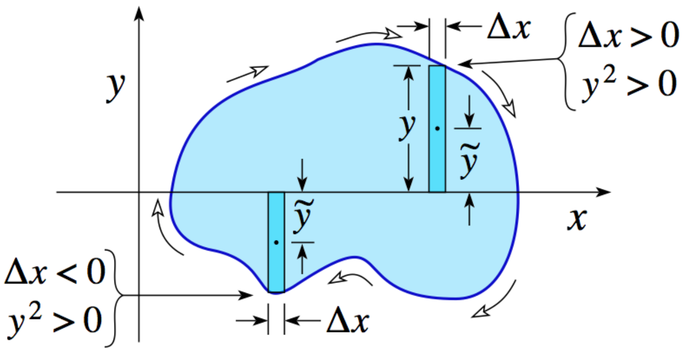
$$\bar{x} = \frac{Q_y}{A} \qquad \bar{y} = \frac{Q_x}{A}$$

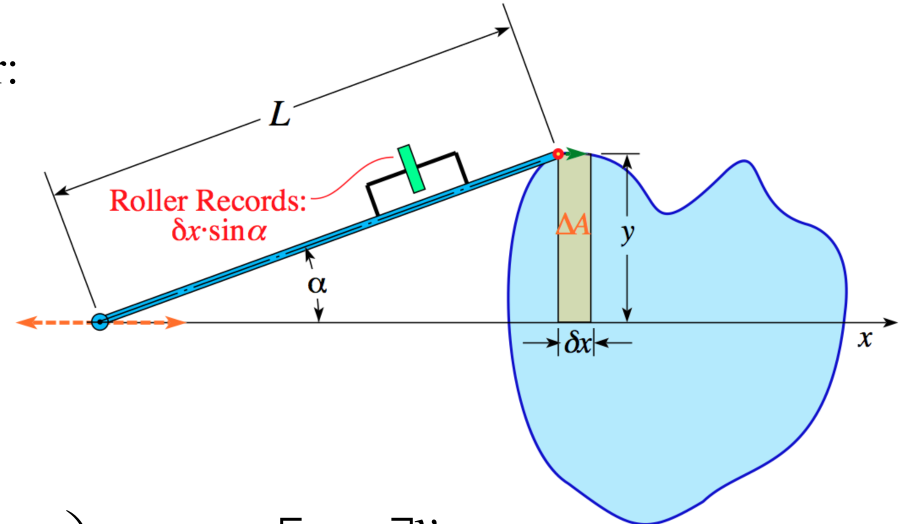
For Example, for Q_x :



Mechanical Determination of the First Moment of Area

Starting from our Linear Planimeter:





$$Q_x = \int_A y \cdot dA = \int_{x_1}^{x_2} \left(\int_0^y y' \cdot dy' \right) dx = \int_{x_1}^{x_2} \left[\frac{y'^2}{2} \right]_0^y dx = \frac{1}{2} \int_{x_1}^{x_2} y^2(x) \cdot dx$$

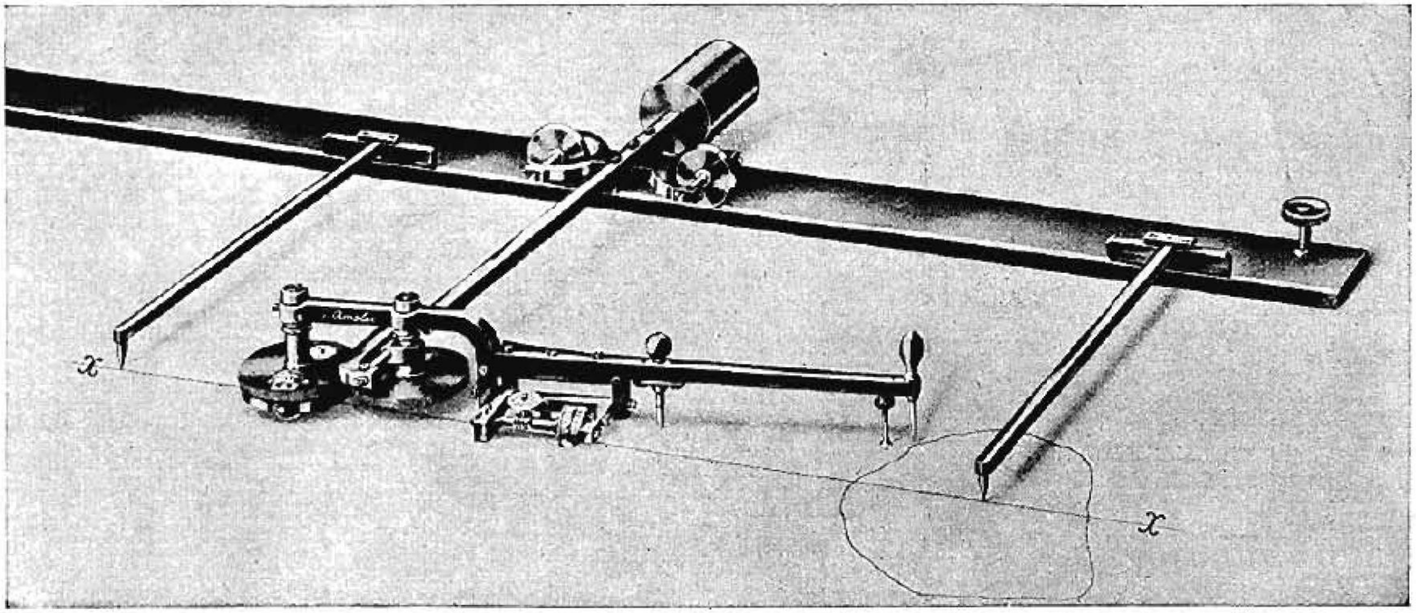
$$y = L \sin\alpha$$

$$Q_x = \frac{1}{2} \oint (L\sin\alpha)^2 dx = \frac{L^2}{2} \oint \frac{\{1-\cos 2\alpha\}}{2} dx = \frac{L^2}{4} \oint dx - \frac{L^2}{4} \oint (\cos 2\alpha) dx$$

$$\left(\oint dx = 0 \right)$$

$$\cos 2\alpha = \sin\left(\frac{\pi}{2} - 2\alpha \right) \quad ; \qquad Q_x = -\frac{L^2}{4} \oint \sin\left(\frac{\pi}{2} - 2\alpha \right) dx$$

Amsler-Integrator Nr. 3 Messing oder Neusilber



Amsler Two-Dial Integrator

Image from the Amsler Catalog.

Offered in the K&E catalog of 1900 as 4270 (electrum)/4272 (brass) at $115/$90.

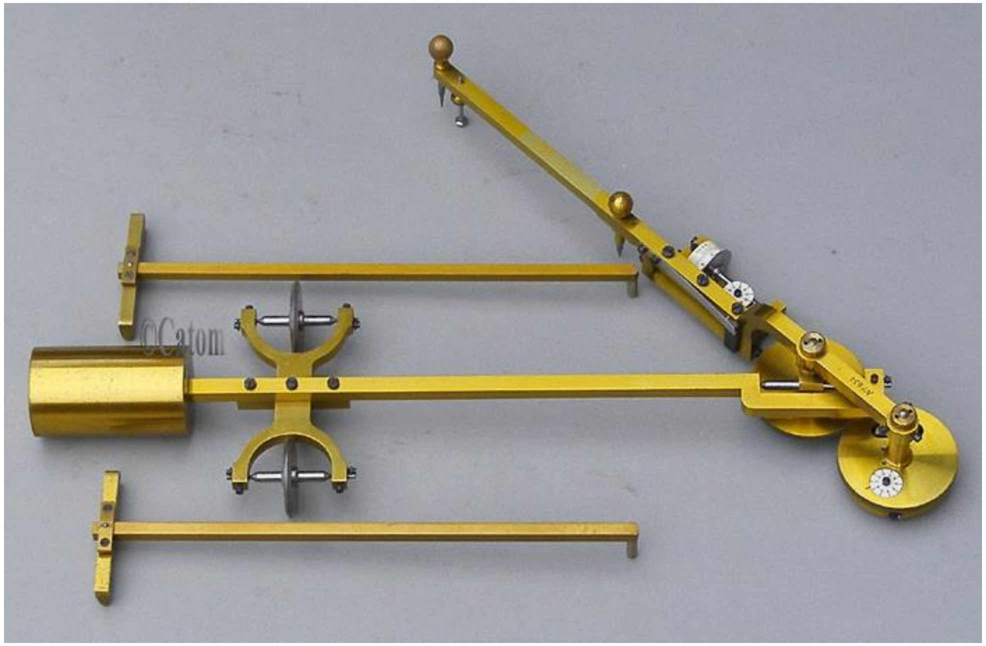
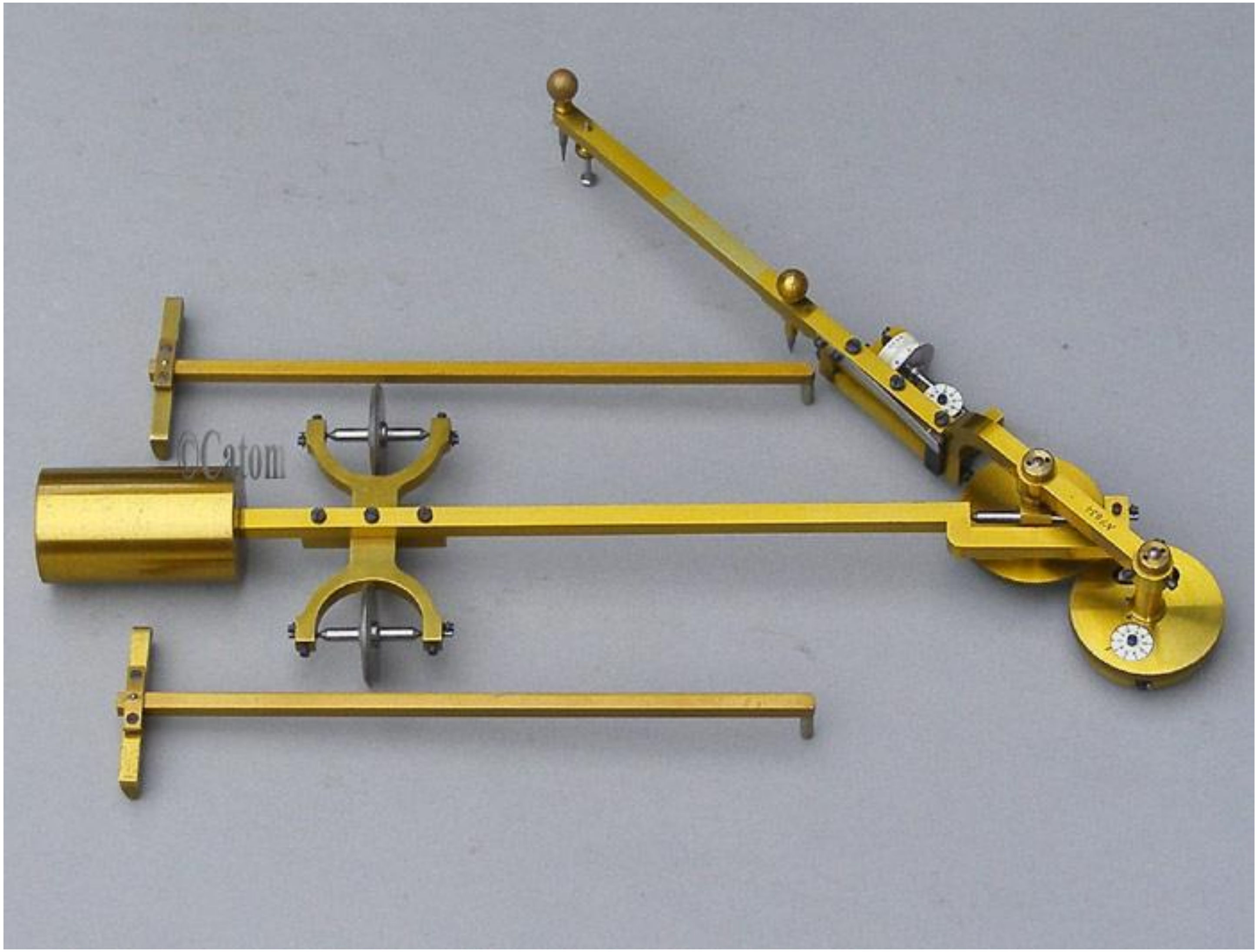


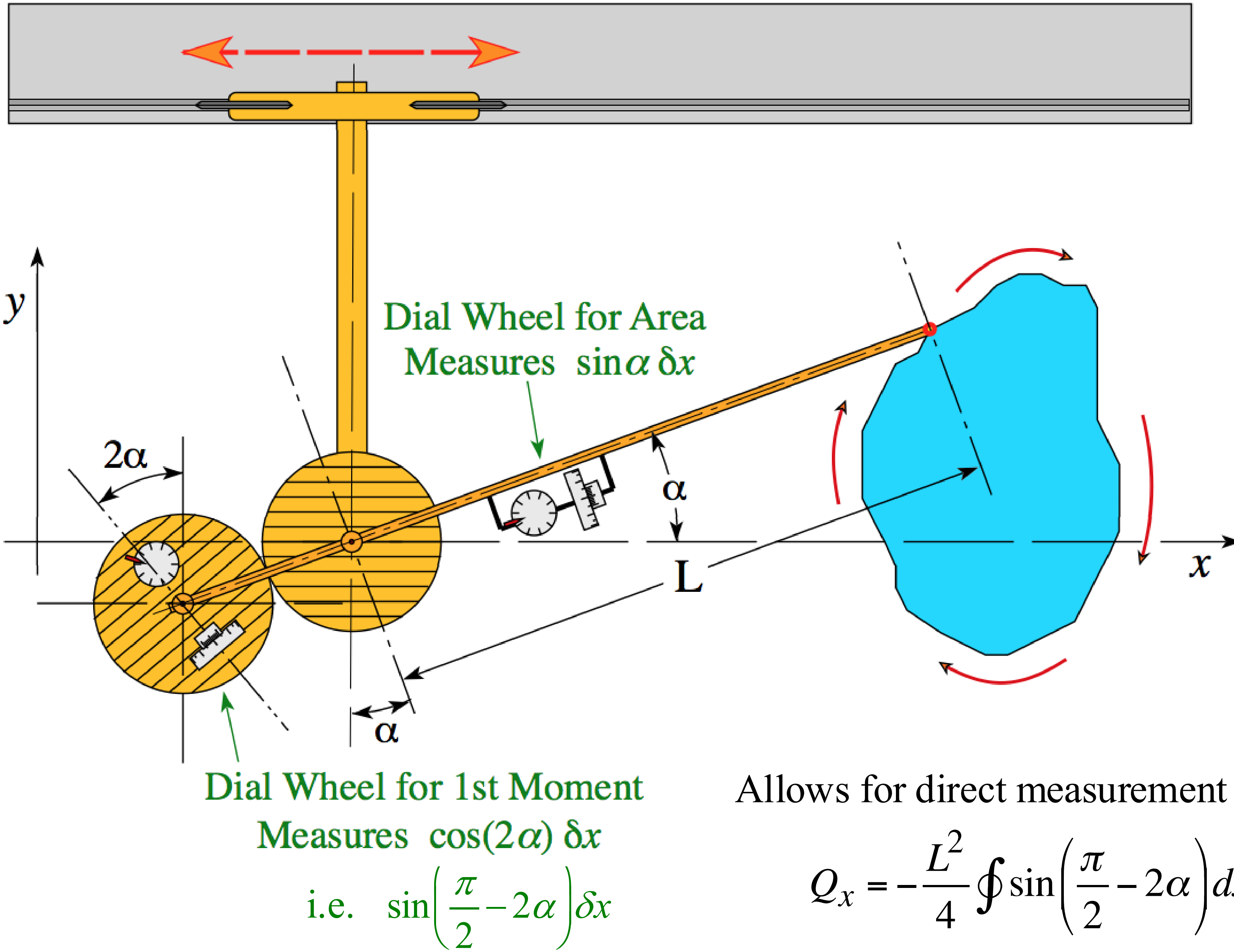
Image from the website of
Fleaglass Antique Scientific Instruments, ...
www.fleaglass.com/index.php?a=2&b=1478
Retrieved 8/27/13.

Amsler Two-Dial Integrator



www.fleaglass.com/index.php?a=2&b=1478

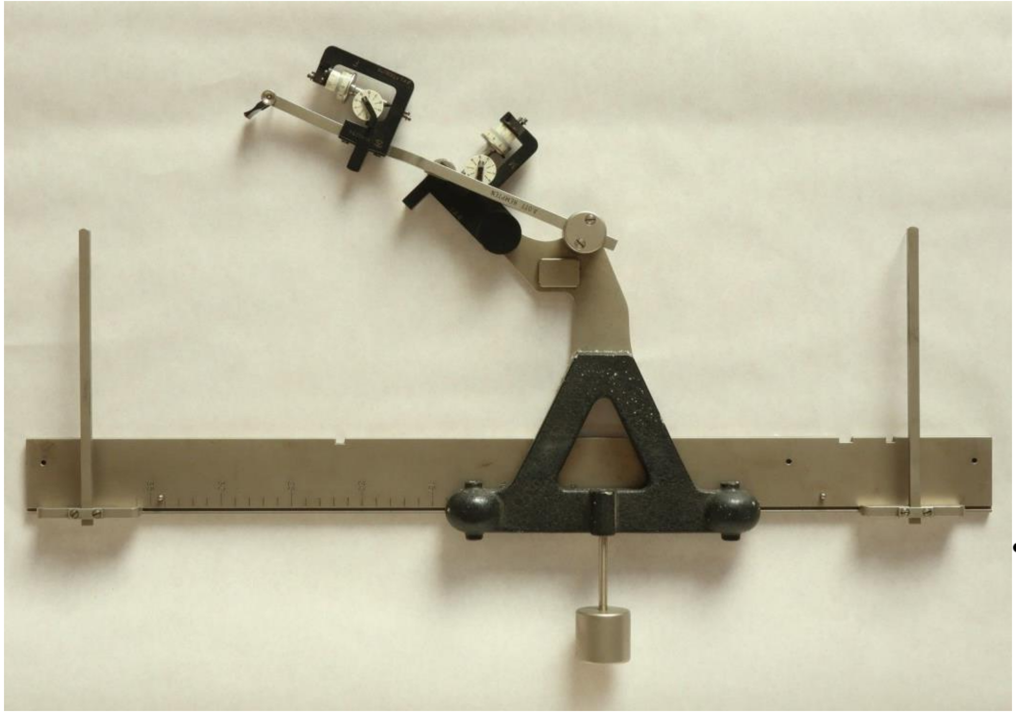
Operation of the Two-Dial Integrator

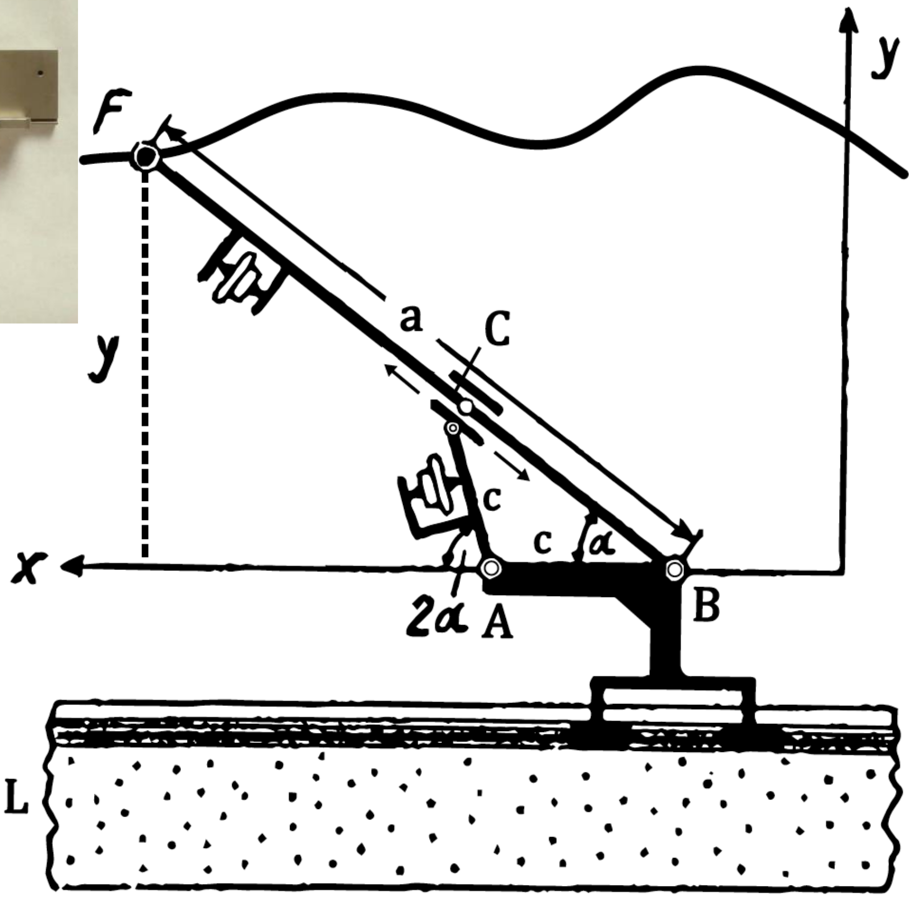


Allows for direct measurement of:

$$Q_x = -\frac{L^2}{4}\oint \sin\left(\frac{\pi}{2}-2\alpha\right)dx$$

More Modern Implementation by Ott



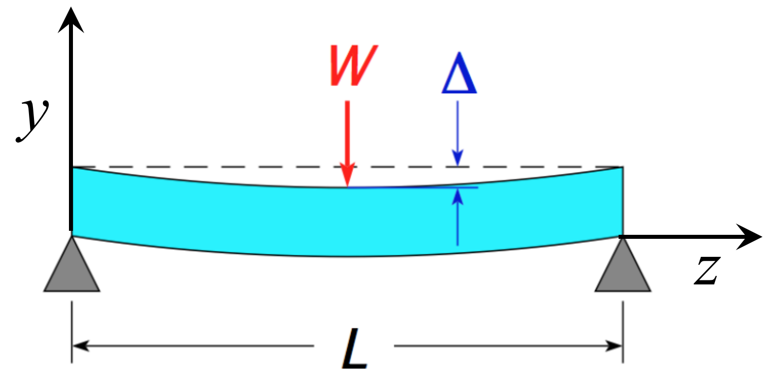


Sketch by Michael Palm. Figure 11 from Historische Integratoren der Firma A. Ott, 2013/2014.

Second Moment or Moment of Inertia?

Classic Example is the Bending of a Beam

L = Length
W = Load at Center
Δ = Deflection at the Middle.
I = Moment of Inertia
E = Young's Modulus
(material stiffness)



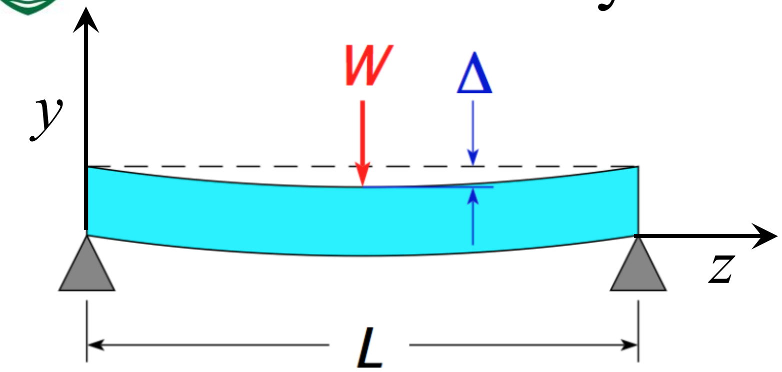
Answer:

$$\Delta = \frac{WL^3}{48EI}$$

where:

$$I = I_{xx} = \frac{1}{3}\oint y^3\, dx$$

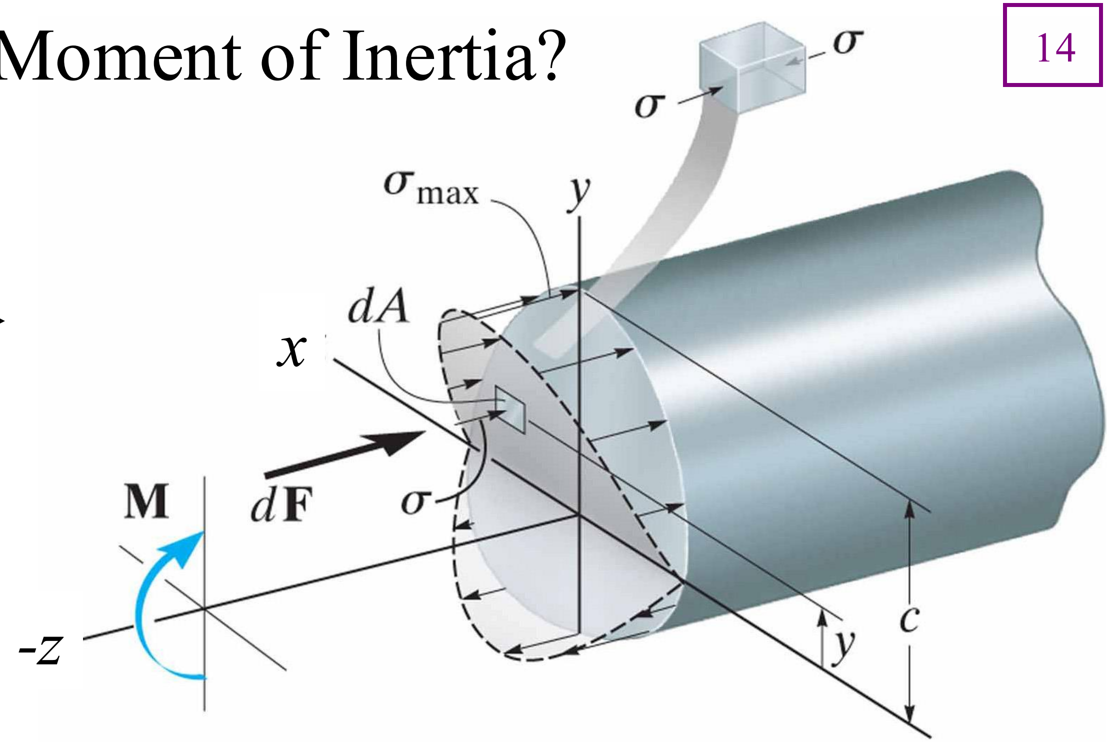
Why the Moment of Inertia?



L = Length
W = Load at Center
Δ = Deflection at the Middle.
I = Moment of Inertia
E = Young's Modulus
(material stiffness)

Answer: $$\Delta = \frac{WL^3}{48EI}$$

$$I = I_{xx} = \frac{1}{3}\oint y^3\, dx$$



Moment is from the sum of force increments ($d\mathbf{F}$) from stress (σ) on increments of area (dA):

$$M = \int_A y\, dF = \int_A y(\sigma dA)$$

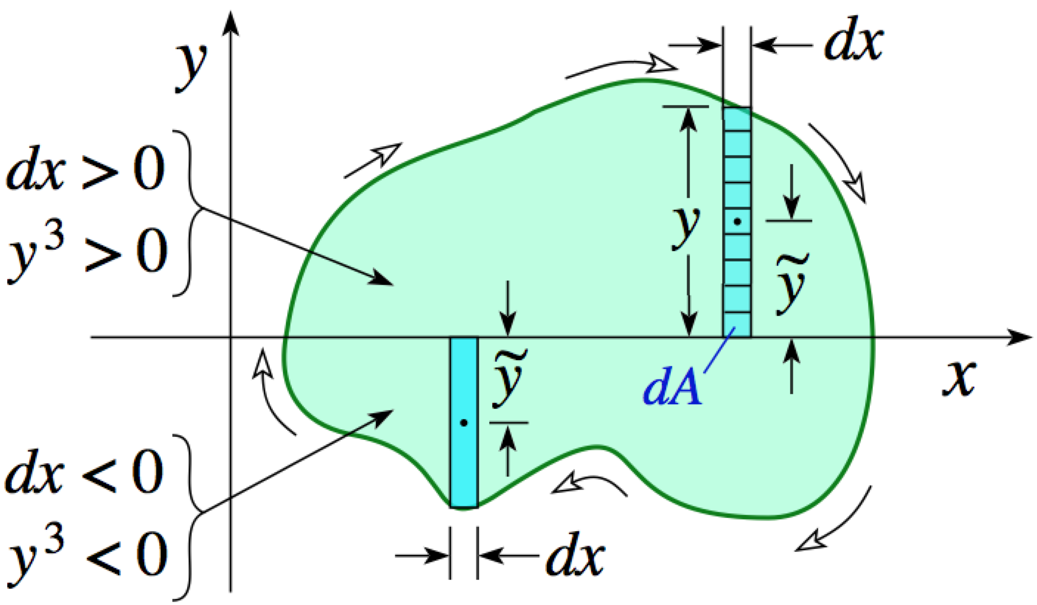
Stress σ is proportional to y.

$$M = \int_A y\left(y\frac{\sigma_{max}}{c}dA\right) = \frac{\sigma_{max}}{c}\int_A y^2\, dA$$

Figure from Hibbeler
(coordinates renamed)

Next Step: Second Moment of Area (Moment of Inertia)

In Beam Bending theory we find the stiffness of a beam (in bending) is proportional to moment of inertia of the cross-section.



For the part $y > 0$:

$$I_{xx} = \int_{x_{min}}^{x_{max}} y^2 dA = \int \left(\int_0^y y'^2 \, dy' \right) dx$$

$$I_{xx} = \frac{1}{3} \int_{x_{min}}^{x_{max}} y^3 \, dx$$

For the entire shape:

$$I_{xx} = \frac{1}{3} \oint y^3 \, dx$$

Contributions to the integral are positive for ($y > 0$; $dx > 0$) and for ($y < 0$; $dx < 0$).

Second Moment of Area
Moment of Inertia

$$I = \iint_A y^2 \, dA$$

$$I = \int_{x_1}^{x_2} \left(\int_0^{y} y'^2 \cdot dy' \right) dx = \int_{x_1}^{x_2} \left[\frac{y'^3}{3} \right]_0^{y} dx = \frac{1}{3} \int_{x_1}^{x_2} y^3(x) \cdot dx$$

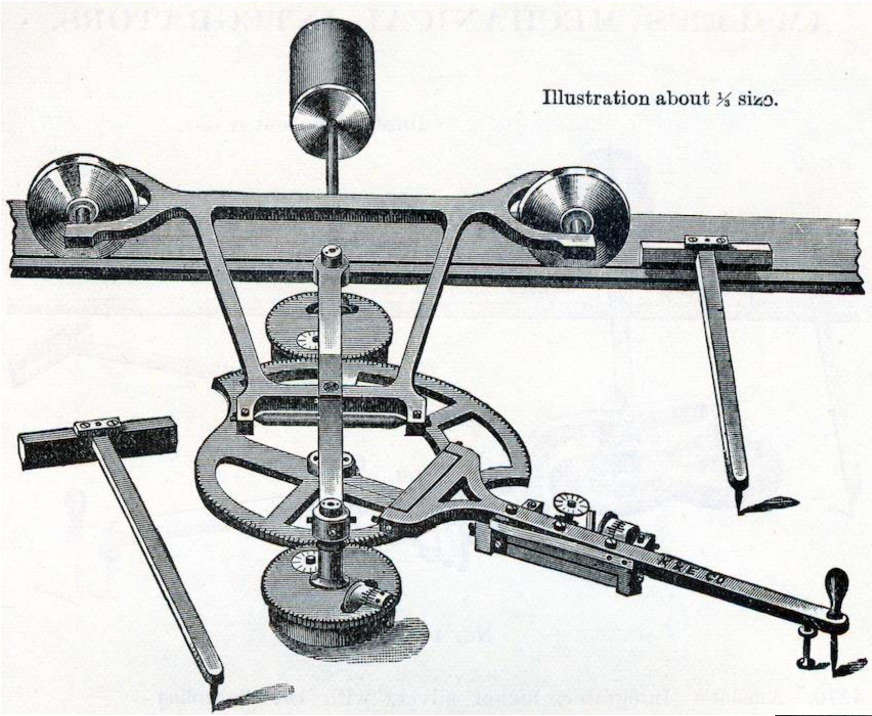
$$I = \frac{1}{3} \oint (L \sin\alpha)^3 dx = \frac{L^3}{3} \oint \left\{ \frac{3}{4} \sin\alpha - \frac{1}{4} \sin 3\alpha \right\} \cdot dx$$

$$I = \frac{L^3}{4} \oint \sin\alpha \, dx - \frac{L^3}{12} \oint \sin(3\alpha) \, dx$$

Can be done with a linear planimeter.

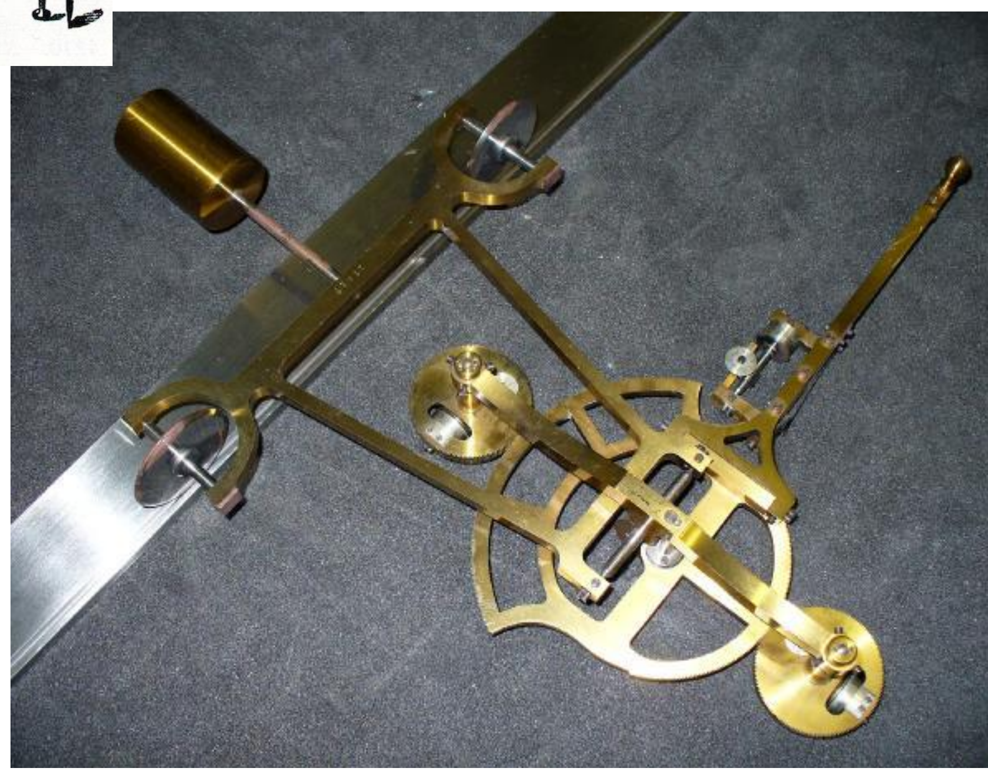
Requires a new gadget.

Three-Dial Amsler Integrator (number 1)



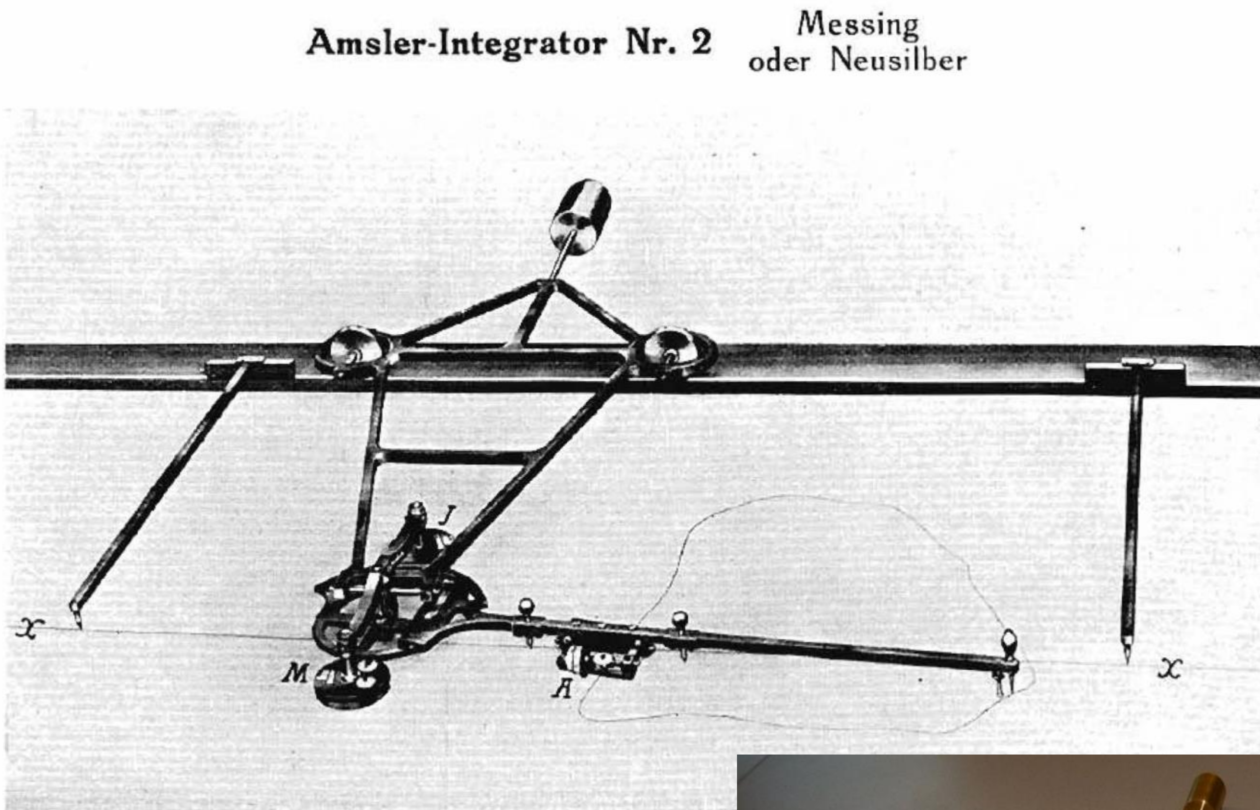
Shown at the Paris 1867 Exhibition

Offered in the K&E catalog of 1900 as 4280 (electrum)/4282 (brass) at $175/$150.



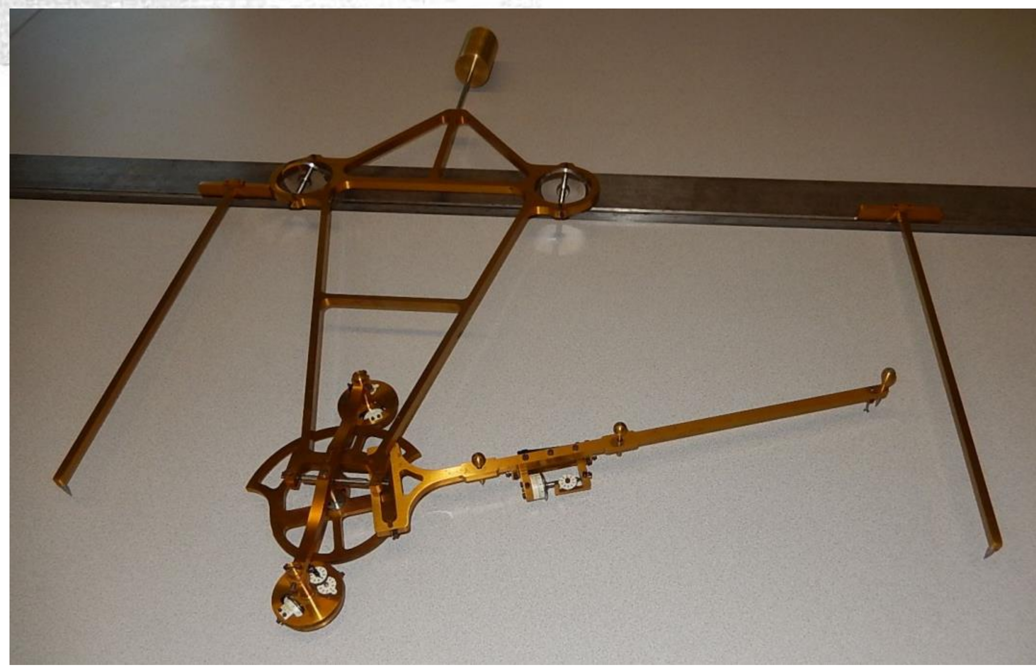
Sketch from Keuffel & Esser Catalog 36th ed.

Photo from www.reunion.iufm.fr/dep/mathematiques/calculsavant/Exposition/ArtsetMetiers/11417-0000-e.html



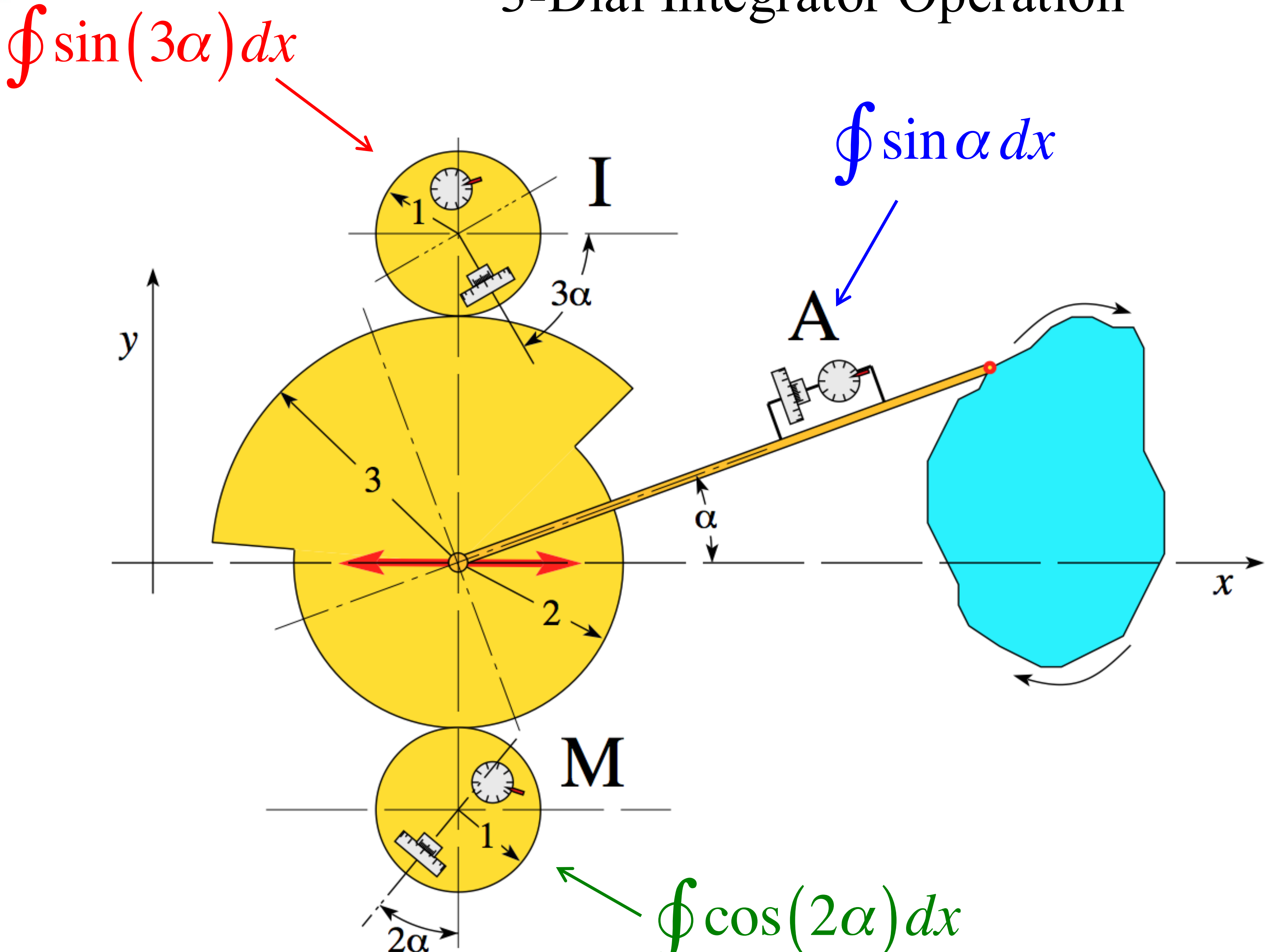
Three-Dial Amsler Integrator Large Version

From Amsler Catalog.

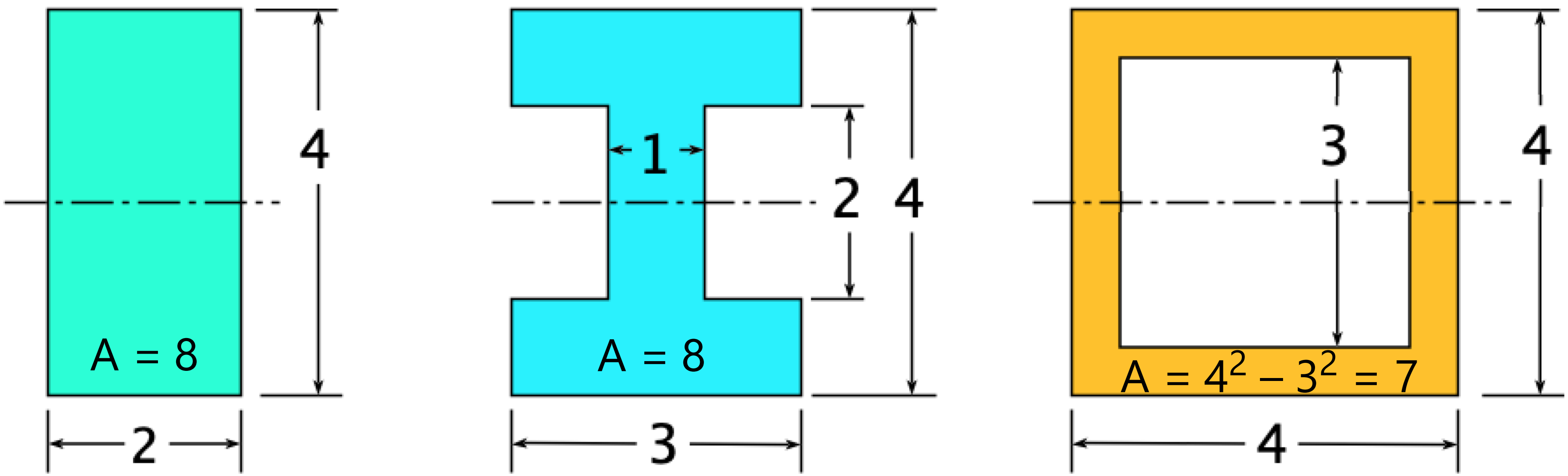


Offered in the K&E catalog of 1900 as 4286 (electrum) or 4288 (brass) at $280 or $230. (Imported by special order.)

3-Dial Integrator Operation



Moment of Inertia in Beam Bending Analysis

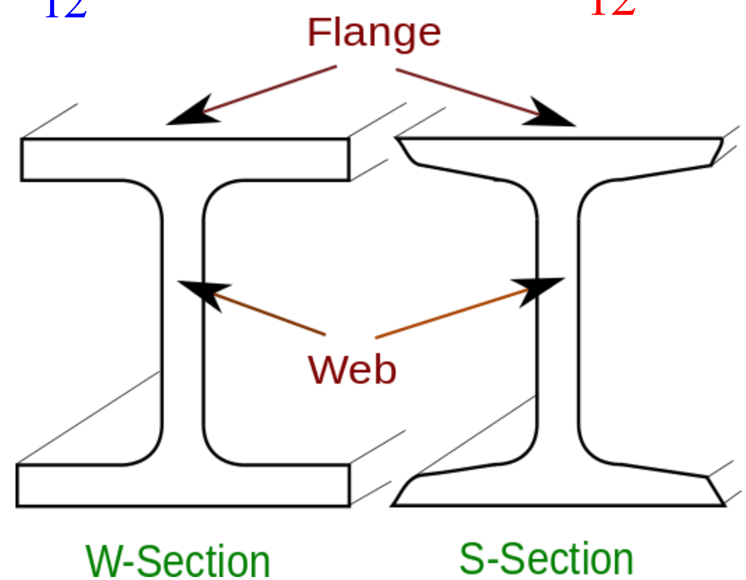


$$I_{xx} = \frac{bh^3}{12}$$

$$I_{xx} = \frac{3\square 4^3}{12} - 2\square\frac{1\square 2^3}{12} = 14.667$$

$$I_{xx} = \frac{4\square 4^3}{12} - \frac{3\square 3^3}{12} = 14.583$$

$$I_{xx} = \frac{bh^3}{12} = \frac{2\square 4^3}{12} = 10.667$$



(Sketch from Wikipedia, *I-beam*, retrieved 2021, author: Bbanerje)

Stability of Floating Bodies

Gravity acts on the boat as a vertical force passing through the center of gravity.

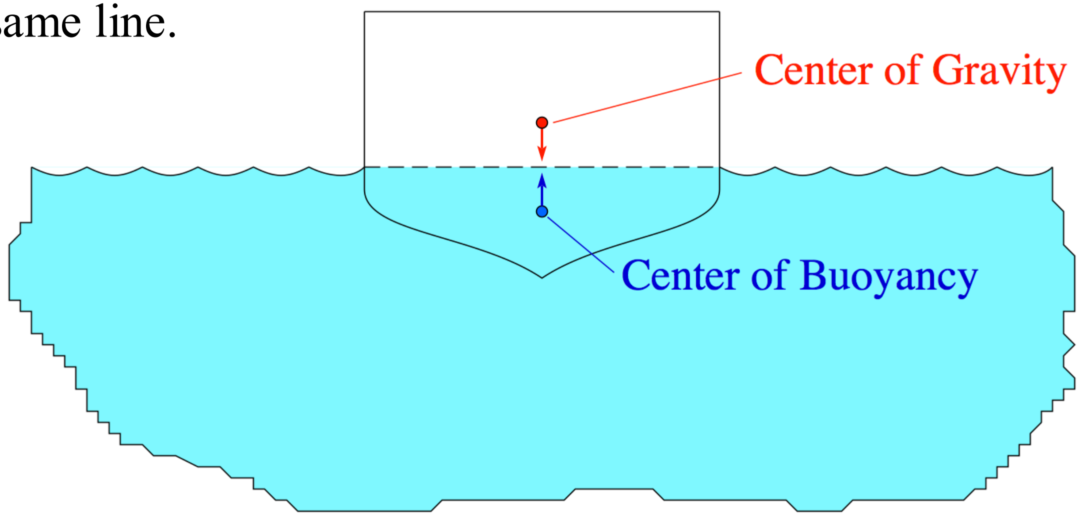
If the boat is to float, there must be a counteracting force equal magnitude and opposite direction, exerted on the boat by the pressure of water on the hull.

This force is called the buoyancy. The magnitude of the buoyancy equals the weight of the water displaced by the hull. That is, the buoyancy equals the volume of boat below the waterline times the density of water.

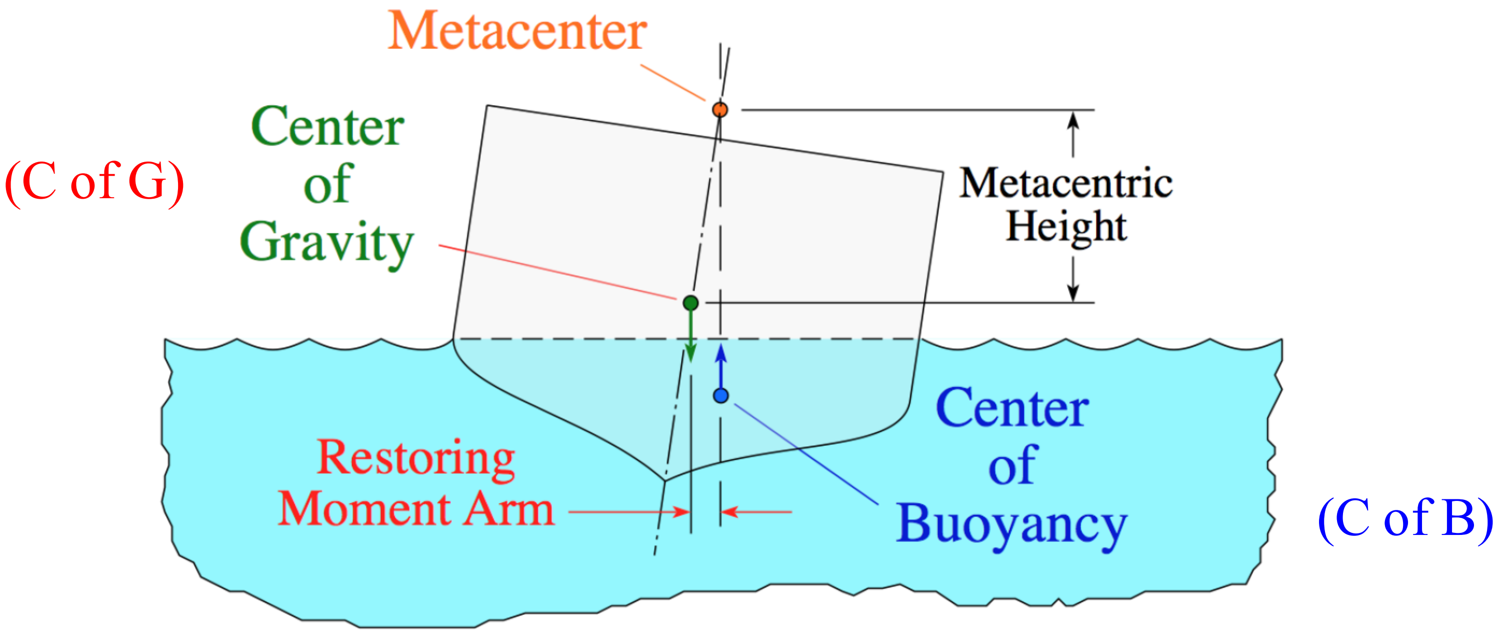
The vector force of the buoyancy acts through the centroid of the displaced volume. The centroid is called the center of buoyancy.

Buoyancy is the force exerted by the displaced water that counteracts the force of gravity of the ship.

If the boat is in static equilibrium, the Gravity force and the buoyancy force are equal and opposite, acting along the same line.



Stability against Perturbation



When a boat is tilted, the center of buoyancy moves.

If the boat is stable, the C of G and the C of B form a couple that restores the boat to its stable, upright position.

That is, stability means the C of B moves farther than to just directly below the C of G.

A line drawn vertically from the C of B intersects the original midline at a point called the Metacenter.

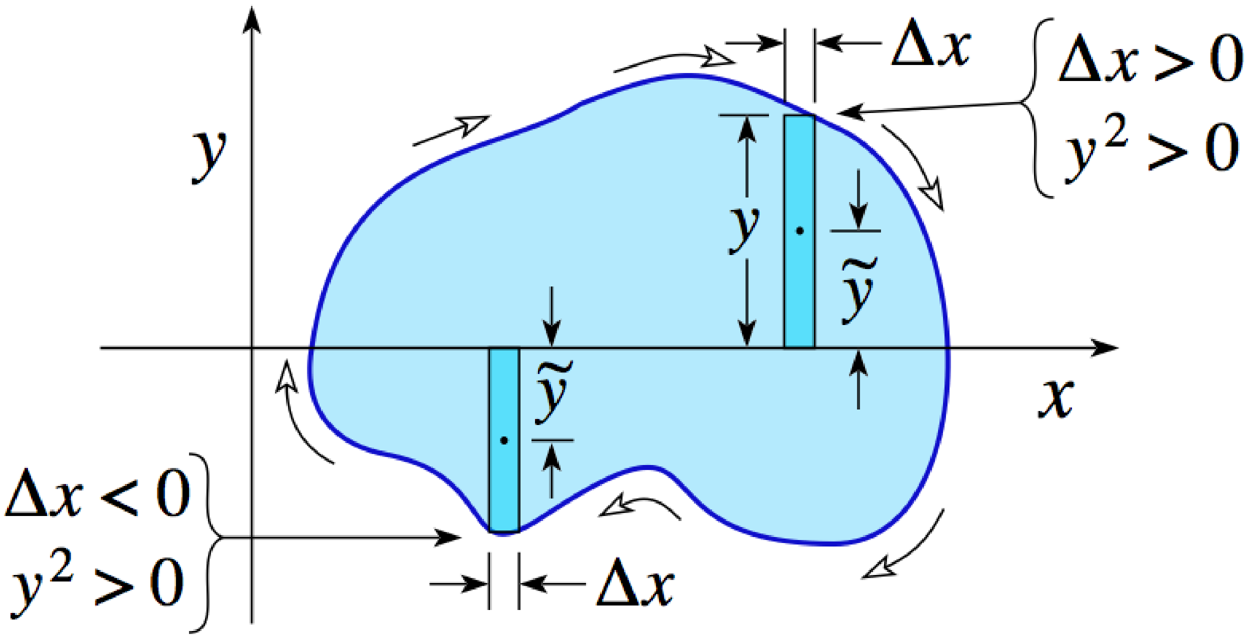
The vertical distance from C of G to Metacenter is the Metacentric Height.

The greater the metacentric height, the greater the stability of the boat.

To find the Center of Buoyancy

- Find the displaced Area: A
- Find the first moment about the horizontal: Q_x .
- Find the first moment about the vertical: Q_y .

$$\bar{y} = \frac{Q_x}{A} \quad ; \quad \bar{x} = \frac{Q_y}{A}$$



Amsler Two-Dial Integrator

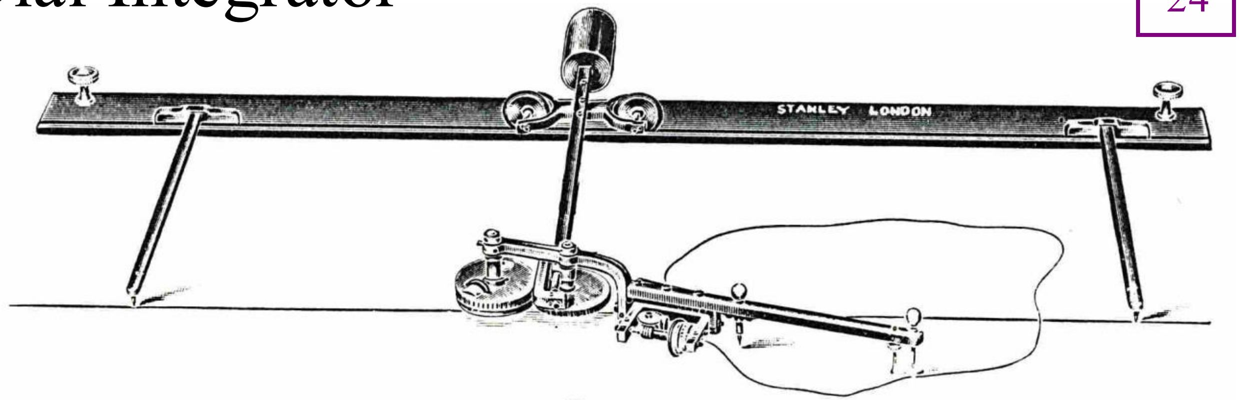


FIG. 9.

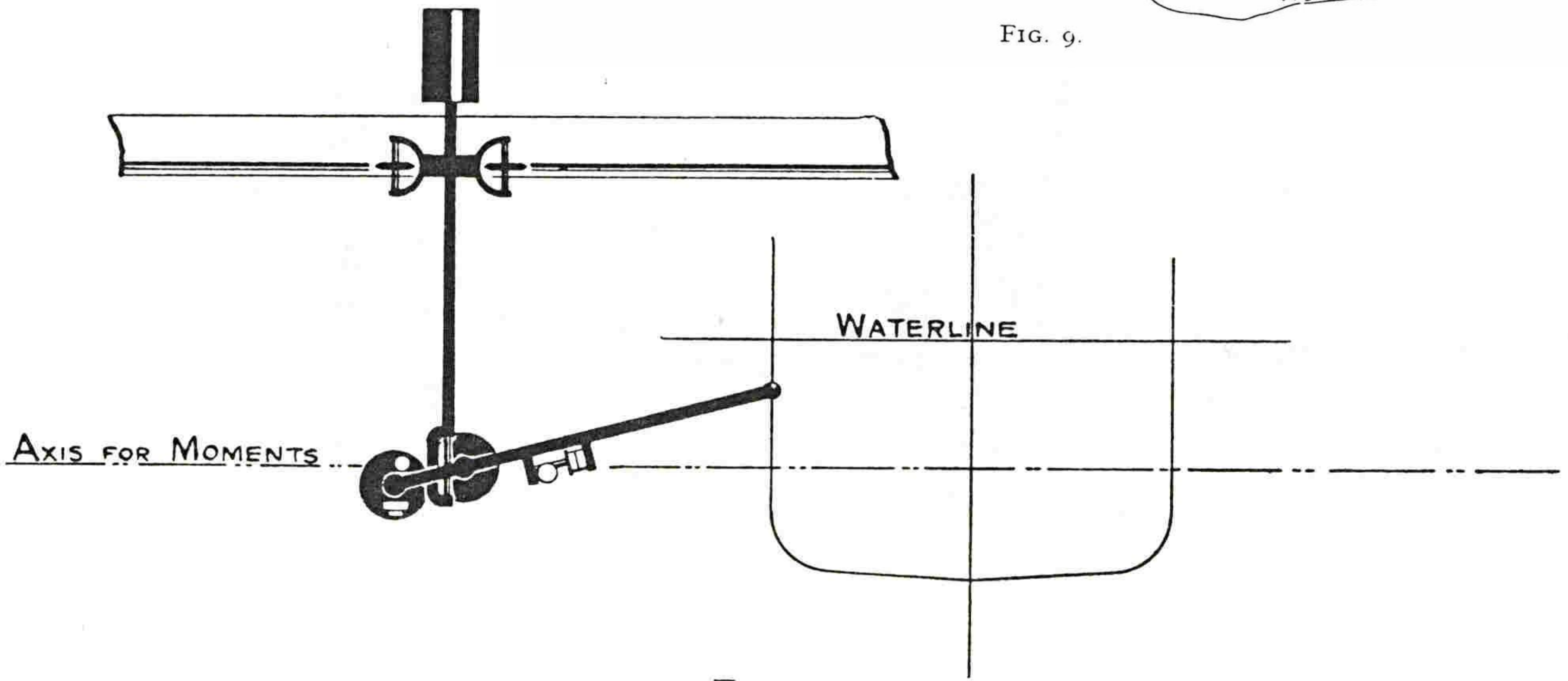


FIG. 13.

A. M. Robb, "The Use of Mechanical Integrating Machines in Naval Architecture"
in Handbook of the Napier Tercentenary Celebration or Modern Instruments and Methods of Calculation.
edited by E.M. Horsburgh, originally 1914, reprinted 1982.

Curves of Stability

The greater the angle of tilt (θ), the farther the C of B moves, and the greater the restoring-force couple.

The plot of restoring couple vs. θ is called the Stability Curve.

The stability curve calculation required locating the C of B for various angles.

For simplicity we assume that the C of G stays fixed in the structure of the ship. This is not true if there are loose items or liquids that can shift position when the ship tilts. (In sailing warships it was rather important to tie down the guns very securely.)

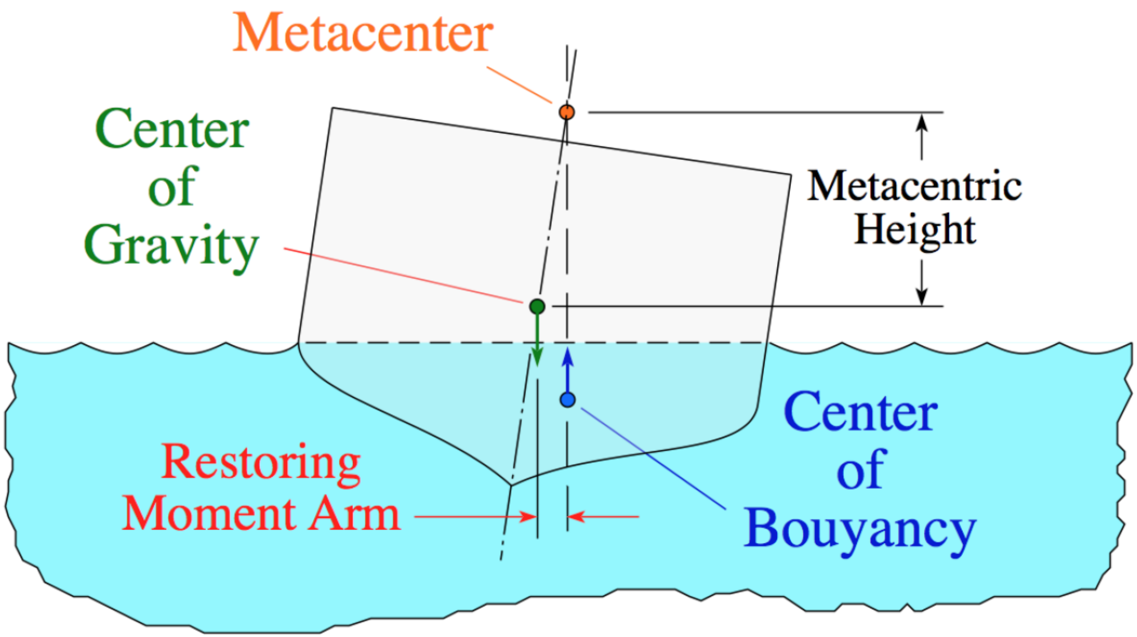
For small angles the ship rotates about the midpoint of the waterline.
For large angle rotations, the new waterline does not cross the midline at the same location as for the upright position.

In the general case, the stability curve is different for different degrees of loading. That is, for different initial waterlines.

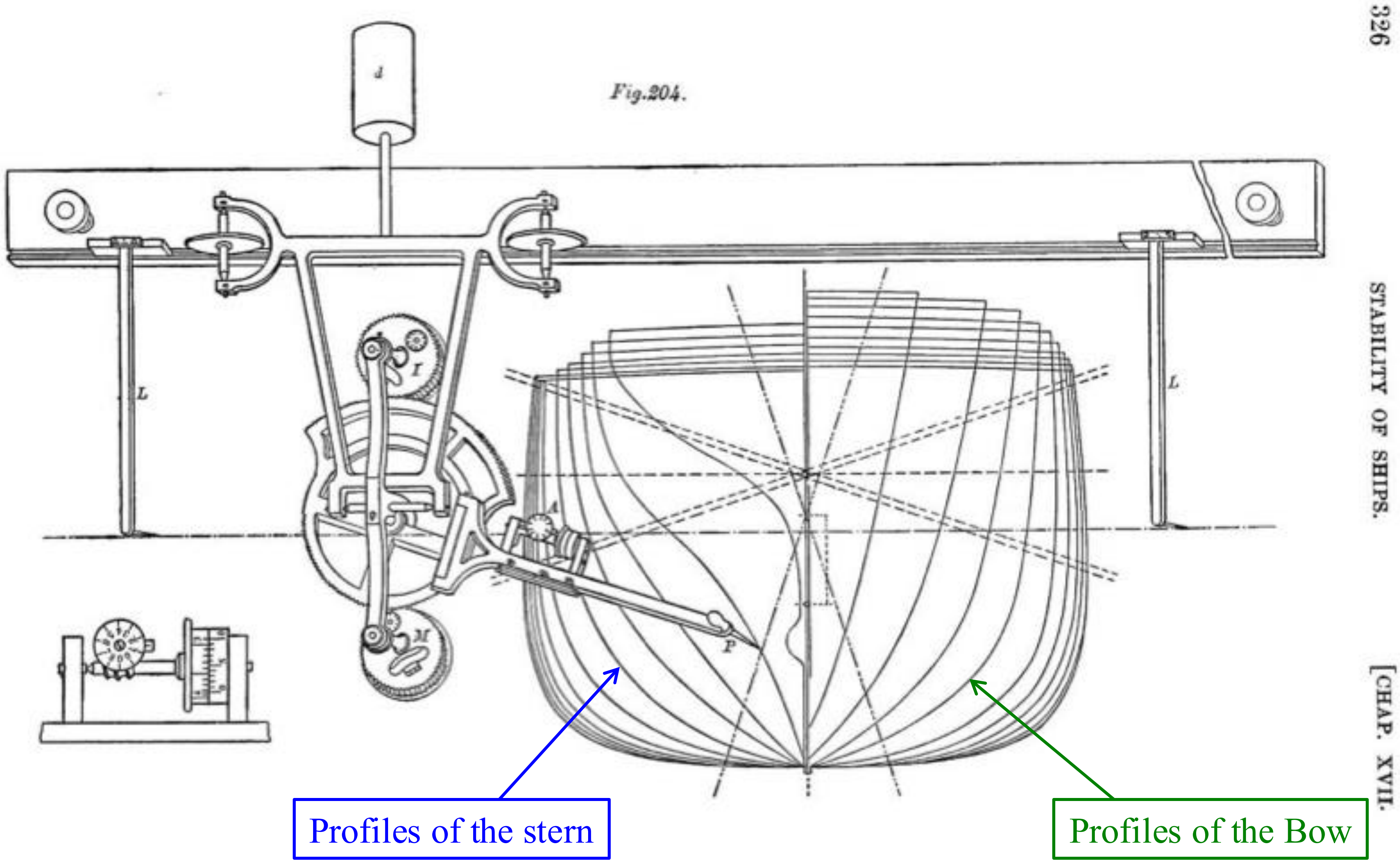
$\Rightarrow$ To ensure general overall stability, several water levels must be considered.

Restoring Force

Restoring Moment is proportional to the distance the C of B has moved, which is the length of the Restoring Moment Arm.



Sketch of Integrator Use.



A Treatise on the Stability of Ships, by Sir Edward J. Reed, K.C.B., F.R.S., M.P., 1885

Determination of Curves of Stability

Calculation of Restoring Moment Arm is repeated for different angles of heel and different water levels.

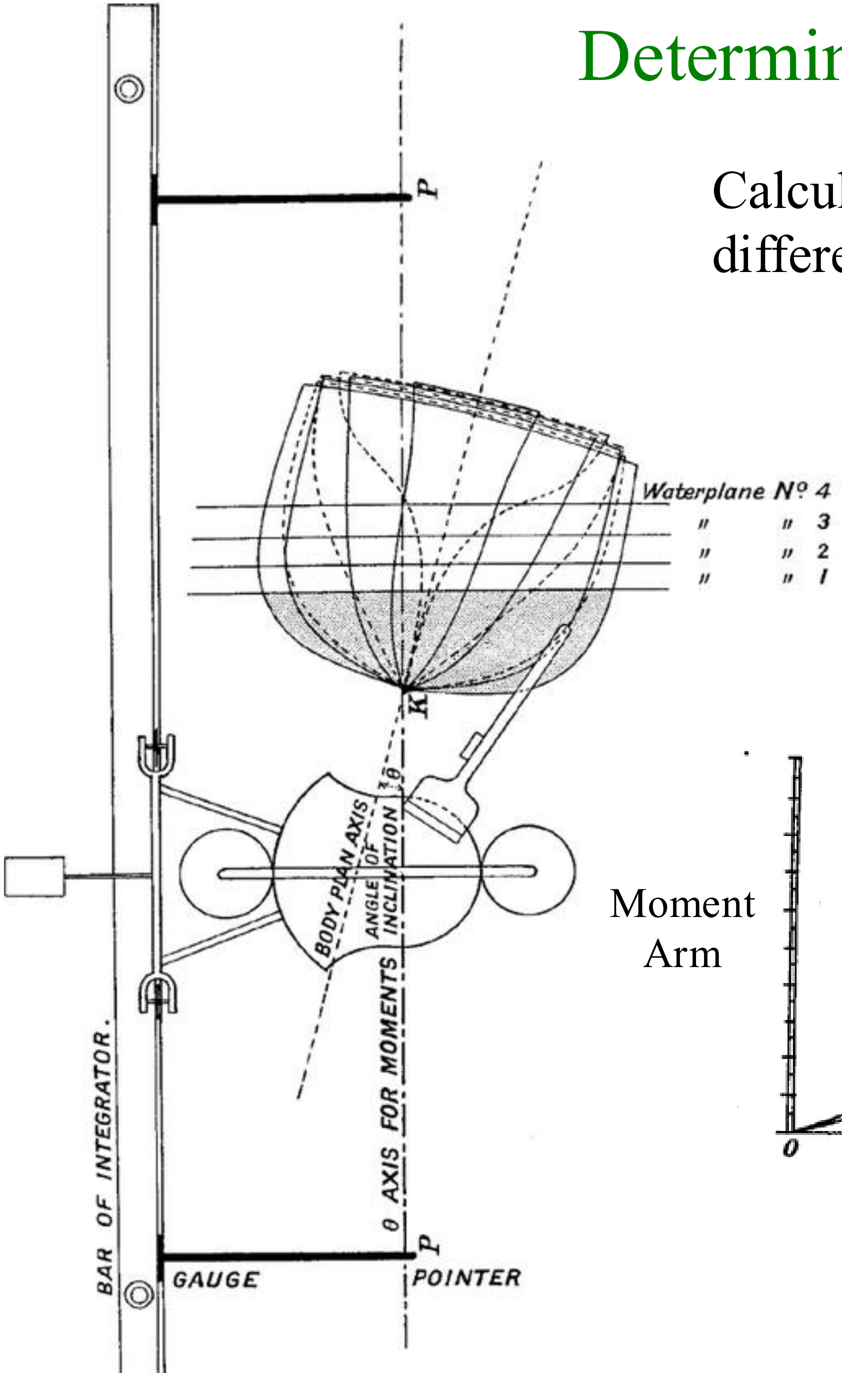


FIG. 44.

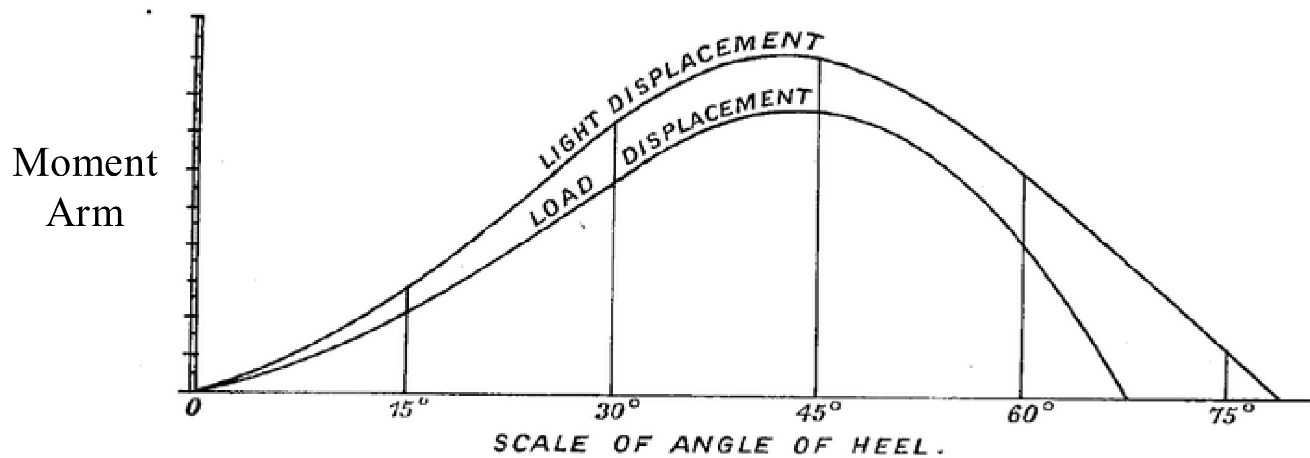
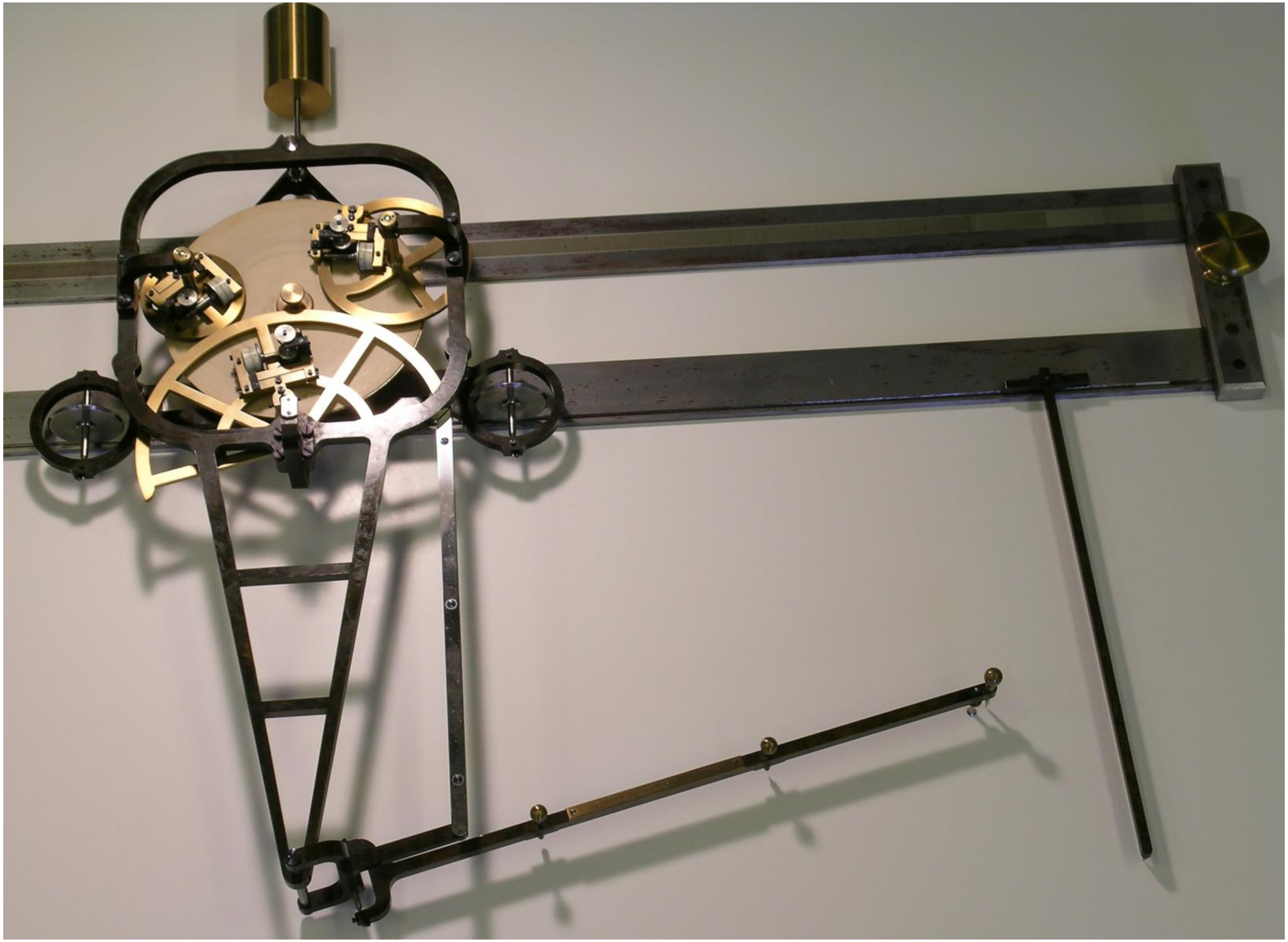


FIG. 47.

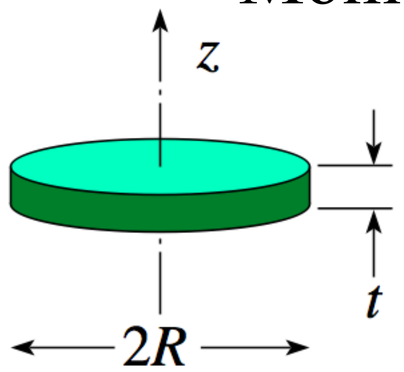
(J.H. Biles, The Design and Construction of Ships, 2nd Edition, 1923, Part IV. Stability, Chapter 5)

The Dartmouth Amsler Integrator



Uses a central rotating disc to drive the three measuring wheels.

Moment of Inertia of Solids of Revolution

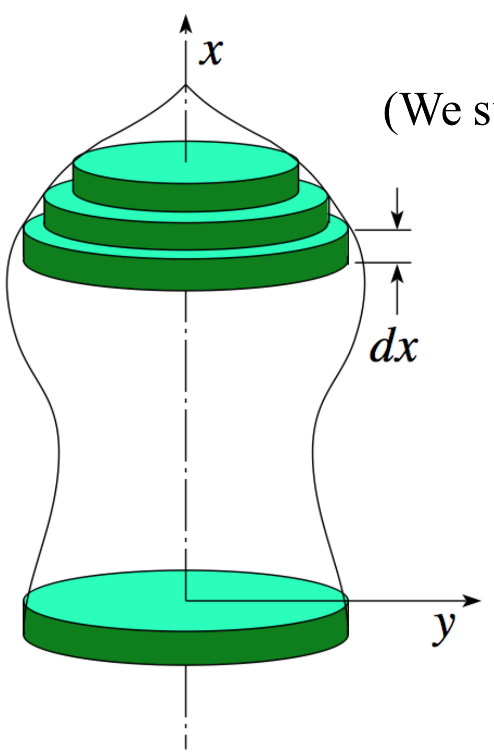


Rotational moment of inertia of a disk: $$J_z = \frac{\pi}{2}\rho R^4 t$$

R = Radius
t = Thickness
ρ = Density

A solid of revolution can be modeled as a stack of disks, each of thickness δx, and radius y.

(We switch coordinate labels so as to match previous integrals.)

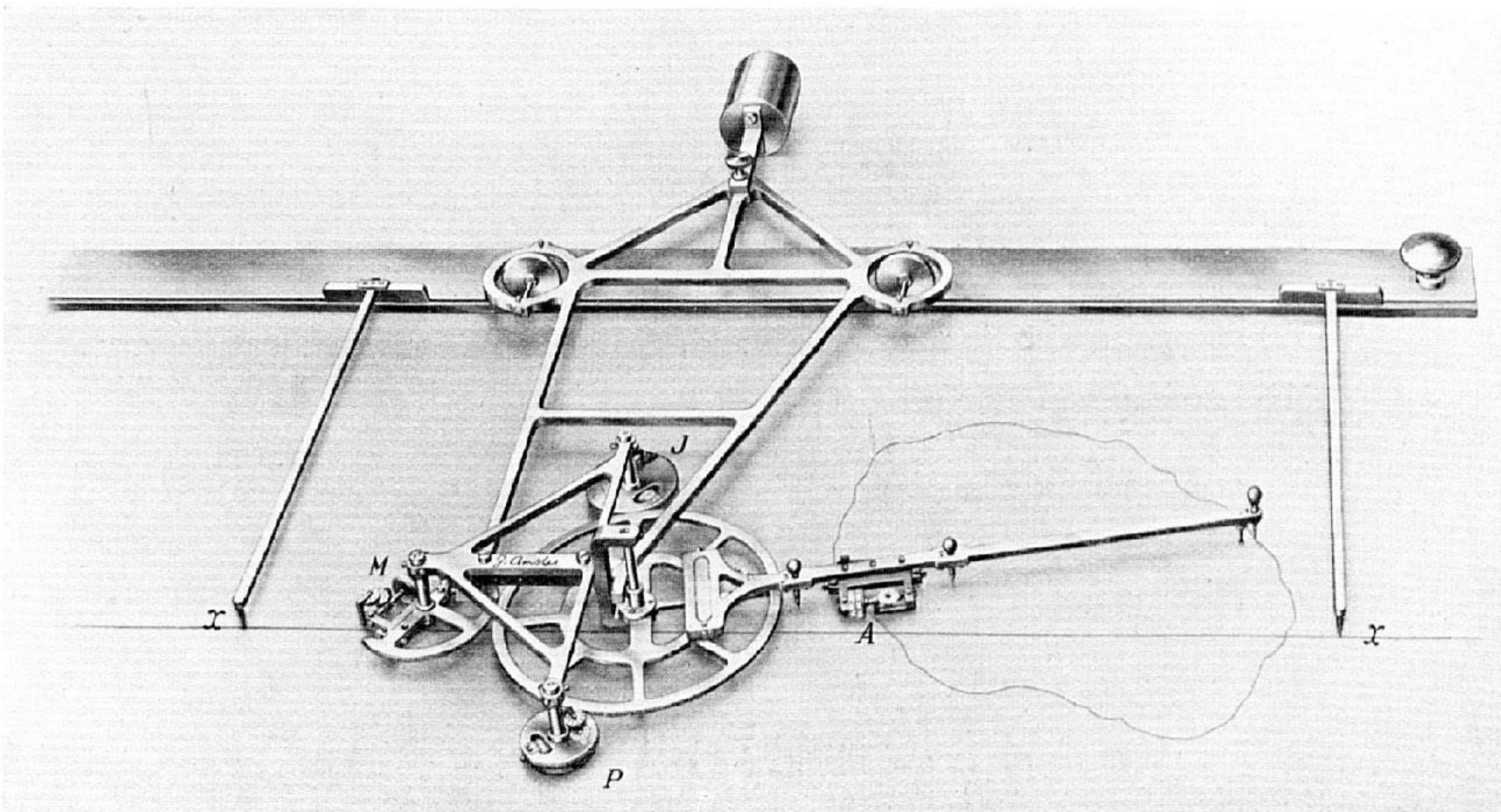


$$J_x = \frac{\pi\rho}{2} \int y^4 \, dx$$

This is the Third Moment of Area.

Amsler's 4-Dial Integrator

Amsler-Integrator Nr. 4 Messing oder Neusilber



Note switch in notation (I → J) for the second moment.

(from Amsler's Catalog)

Third Moment of Area

$$P = \int_A y^3 dA = \frac{1}{4}\oint y^4\, dx = \frac{L^4}{4}\oint \sin^4\alpha\, dx$$

Use: $\sin^2\alpha = \dfrac{1-\cos(2\alpha)}{2}$

$$P = \frac{L^4}{4}\oint \left(\frac{1-\cos(2\alpha)}{2}\right)^2 dx = \frac{L^4}{16}\oint \left(1 - 2\cos(2\alpha) + \cos^2(2\alpha)\right) dx$$

Use: $\cos^2(2\alpha) = \dfrac{1+\cos(4\alpha)}{2}$

$$P = \frac{L^4}{32}\oint \left(2 - 4\cos(2\alpha) + 1 + \cos(4\alpha)\right) dx$$

$$\oint dx = 0$$

$$P = \frac{L^4}{32}\oint \left(-4\cos(2\alpha) + \cos(4\alpha)\right) dx$$

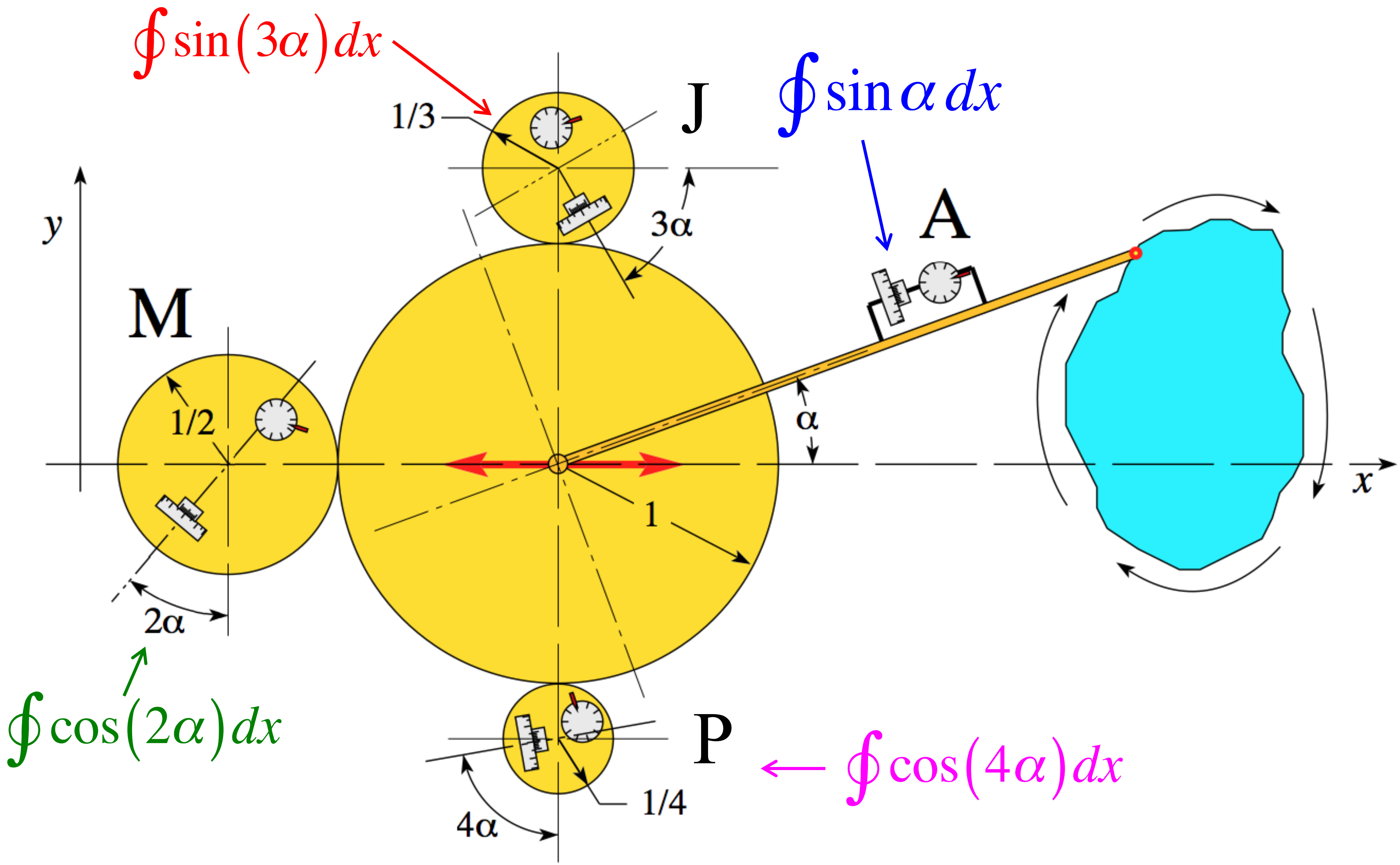
$\Rightarrow$ We need a gadget that measures $\cos(2\alpha)$ and $\cos(4\alpha)$.

Amsler's 4-Dial Integrator

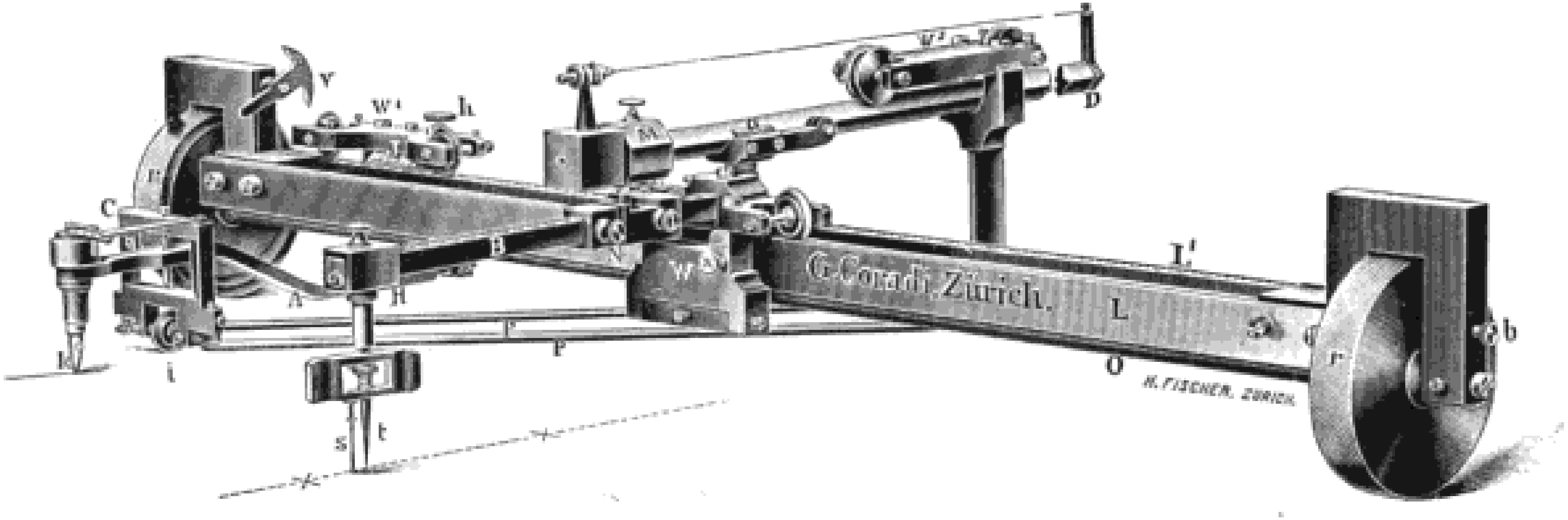
(as from Harvard's museum of scientific instruments.)

(www.dssmhi1.fas.harvard.edu/emuseumdev, 2013)

4-Dial Integrator Operation



Graphical Integration



No. 41

Integraph – from the 1915 Coradi catalog

(shown on Wikipedia, *Integraph*, 2022)

Summary for Amsler-type Integrators

- Areas and Moments of Areas.
- Mechanism of the Amsler Polar Planimeter.
- Amsler 2-Dial Integrator.
- Amsler 3-Dial Integrator.
- Amsler 4-Dial Integrator.
- Graphical Integration – Coradi Integraph.

We have not finished with mechanical integrators. They were used in harmonic analyzers and analog computing devices.

Third Moment of Area in Naval Architecture

Explain how the third moment of area was used to calculate the dynamic stability of ships.

Integraph

Bruno Abdank-Abakanowicz
(1852 – 1900)
Polish-Lithuanian Inventor
and Electrical Engineer

(Wikipedia, *Bruno Abakanowicz*, 2022)